1 Introduction (0 pts)

The Algebra of Noncommutative Operators: In this power round, we will consider the algebra of noncommutative operators. We will define operators as *any objects* that satisfy the properties below. We will use bold-faced, upper case letters, such as $\mathbf{A}$, to denote operators and lower case letters to denote complex numbers. We can add operators, multiply two operators, and multiply operators by numbers. Almost all of the properties that we take for granted for real numbers hold for operators. The only exception is that $\mathbf{AB} \neq \mathbf{BA}$ where $\mathbf{A}, \mathbf{B}$ are operators. Also you cannot add a number and an operator. In particular, the following properties hold:

Suppose that $a, b, ...$ are complex numbers and $\mathbf{A}, \mathbf{B}, ...$ are operators. You may use the following properties without explicitly stating them:

(a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

(c) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.

(d) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$, $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.

(e) $(a + b)\mathbf{C} = a\mathbf{C} + b\mathbf{C}$.

(f) $a(\mathbf{B} + \mathbf{C}) = a\mathbf{B} + a\mathbf{C}$

(g) $a(b\mathbf{C}) = (ab)\mathbf{C}$.

(h) $\mathbf{A}(b\mathbf{C}) = b(\mathbf{AC}) = (b\mathbf{A})\mathbf{C}$.

(i) There is a zero operator, often written as $\mathbf{0}$ such that $\mathbf{0} + \mathbf{A} = \mathbf{A}$ for all operators $\mathbf{A}$.

(j) $0\mathbf{A} = \mathbf{0A} = \mathbf{A0} = \mathbf{0}$ (where $\mathbf{0}$ is the zero operator and 0 is the complex number zero).

(k) Given a operator, $\mathbf{A}$, there exists another operator, denoted by $-\mathbf{A}$ such that $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$.

(l) There exists a multiplicative identity for operators called the identity operator, often written as $\mathbf{I}$. The identity operator is such that $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$ for all operators $\mathbf{A}$.

2 Manipulating Commutators (25 pts)

Commutator: As discussed above, it is not generally true that $\mathbf{AB} = \mathbf{BA}$ for operators. Therefore, it can be useful to consider the quantity, $\mathbf{AB} - \mathbf{BA}$, which is not generally zero. This quantity is called the commutator of $\mathbf{A}$ and $\mathbf{B}$ and is written as follows

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA} \tag{1}$$

P1 (a) **(2 pts)** Prove that $[\mathbf{A}, \mathbf{A}] = \mathbf{0}$ and $[\mathbf{A}, \mathbf{B}] = -[\mathbf{B}, \mathbf{A}]$.

(b) **(2 pts)** Prove the following two properties of commutators

$$[\mathbf{A}, \mathbf{BC}] = \mathbf{B}[\mathbf{A}, \mathbf{C}] + [\mathbf{A}, \mathbf{B}]\mathbf{C} \qquad [\mathbf{A}, b\mathbf{B} + c\mathbf{C}] = b[\mathbf{A}, \mathbf{B}] + c[\mathbf{A}, \mathbf{C}] \tag{2}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are operators and a, b are numbers.

(c) **(2 pts)** Prove the Jacobi Identity:

$$[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] + [\mathbf{B}, [\mathbf{C}, \mathbf{A}]] + [\mathbf{C}, [\mathbf{A}, \mathbf{B}]] = \mathbf{0} \tag{3}$$

Solution:

(a) $[\mathbf{A}, \mathbf{A}] = \mathbf{AA} - \mathbf{AA} = \mathbf{0}$ and $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA} = -(\mathbf{BA} - \mathbf{AB}) = -[\mathbf{B}, \mathbf{A}]$.

(b) $\mathbf{B}[\mathbf{A}, \mathbf{C}] + [\mathbf{A}, \mathbf{B}]\mathbf{C} = \mathbf{B}(\mathbf{AC} - \mathbf{CA}) + (\mathbf{AB} - \mathbf{BA})\mathbf{C} = \mathbf{ABC} - \mathbf{BCA} = [\mathbf{A}, \mathbf{BC}]$ and $b[\mathbf{A}, \mathbf{B}] + c[\mathbf{A}, \mathbf{C}] = b(\mathbf{AB} - \mathbf{BA}) + c(\mathbf{AC} - \mathbf{CA}) = \mathbf{A}(b\mathbf{B} + c\mathbf{C}) - (b\mathbf{B} + c\mathbf{C})\mathbf{A} = [\mathbf{A}, b\mathbf{B} + c\mathbf{C}]$

(c) $[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] = \mathbf{ABC} - \mathbf{ACB} - \mathbf{BCA} + \mathbf{CBA} = (\mathbf{ABC} - \mathbf{BCA}) + (\mathbf{CBA} - \mathbf{ACB})$. Summing cyclically gives the result.

P2 **(3 pts)** Suppose that $[\mathbf{A}, \mathbf{B}] = \mathbf{0}$. If n is a positive integer, prove that $[\mathbf{A}^n, \mathbf{B}] = \mathbf{0}$.

Solution: Proceed by induction. The base case, $n = 1$, is given. Then using the inductive hypothesis, we get

$$[\mathbf{A}^n, \mathbf{B}] = [\mathbf{A}^{n-1}, \mathbf{B}]\mathbf{A} + \mathbf{A}^{n-1}[\mathbf{A}, \mathbf{B}] = \mathbf{0} \tag{4}$$

For your later reference, it is also true that

$$[\mathbf{AB}, \mathbf{C}] = \mathbf{A}[\mathbf{B}, \mathbf{C}] + [\mathbf{A}, \mathbf{C}]\mathbf{B} \qquad [a\mathbf{A} + b\mathbf{B}, \mathbf{C}] = a[\mathbf{A}, \mathbf{C}] + b[\mathbf{B}, \mathbf{C}] \tag{5}$$

Angular Momentum Algebra: An ordered triple of three operators, $(\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3) = \vec{\mathbf{J}}$, are said to form an angular momenum algebra if

$$[\mathbf{J}_1, \mathbf{J}_2] = i\mathbf{J}_3 \qquad [\mathbf{J}_2, \mathbf{J}_3] = i\mathbf{J}_1 \qquad [\mathbf{J}_3, \mathbf{J}_1] = i\mathbf{J}_2 \tag{6}$$

Note $i = \sqrt{-1}$ is the imaginary unit.

Later in the power round, it will be useful to write the first condition as $[\mathbf{J}_i, \mathbf{J}_j] = i\sum_{k=1}^{3} \epsilon_{i,j,k}\mathbf{J}_k$ for $i = 1, 2, 3$ and $j = 1, 2, 3$ where $\epsilon_{i,j,k}$ is the the Levi-Civita symbol is defined as follows: $\epsilon_{1,2,3} = \epsilon_{2,3,1} = \epsilon_{3,1,2} = 1$ and $\epsilon_{3,2,1} = \epsilon_{2,1,3} = \epsilon_{1,3,2} = -1$. If any of the i, j, k are equal, then $\epsilon_{i,j,k} = 0$.

P3 (a) **(3 pts)** Define $\mathbf{J}_+ = \mathbf{J}_1 + i\mathbf{J}_2$ and $\mathbf{J}_- = \mathbf{J}_1 - i\mathbf{J}_2$. Prove that

$$[\mathbf{J}_3, \mathbf{J}_+] = \mathbf{J}_+ \qquad [\mathbf{J}_3, \mathbf{J}_-] = -\mathbf{J}_- \qquad [\mathbf{J}_+, \mathbf{J}_-] = 2\mathbf{J}_3 \tag{7}$$

(b) **(3 pts)** Define $\mathbf{J}^2 = (\mathbf{J}_1)^2 + (\mathbf{J}_2)^2 + (\mathbf{J}_3)^2$. Prove that $[\mathbf{J}^2, \mathbf{J}_i] = 0$ for $i = 1, 2, 3$.

Solution:

(a) $[\mathbf{J}_3, \mathbf{J}_\pm] = [\mathbf{J}_3, \mathbf{J}_1] \pm i[\mathbf{J}_3, \mathbf{J}_2] = i\mathbf{J}_2 \pm i(-1)i\mathbf{J}_1 = \pm(\mathbf{J}_1 \pm i\mathbf{J}_2)$ which proves the first two identities. Next, $[\mathbf{J}_+, \mathbf{J}_-] = [\mathbf{J}_1 + i\mathbf{J}_2, \mathbf{J}_1 - i\mathbf{J}_2] = [\mathbf{J}_1, \mathbf{J}_1] + (-i)[\mathbf{J}_1, \mathbf{J}_2] + (i)[\mathbf{J}_2, \mathbf{J}_1] + [\mathbf{J}_2, \mathbf{J}_2] = 2\mathbf{J}_3$

(b) By cyclic symmetry, it suffices to consider $i = 1$. In this case, $[\mathbf{J}_1^2 + \mathbf{J}_2^2 + \mathbf{J}_3^2, \mathbf{J}_1] = 0 + [\mathbf{J}_2^2, \mathbf{J}_1] + [\mathbf{J}_3^2, \mathbf{J}_1] = \mathbf{J}_2[\mathbf{J}_2, \mathbf{J}_1] + [\mathbf{J}_2, \mathbf{J}_1]\mathbf{J}_2 + \mathbf{J}_3[\mathbf{J}_3, \mathbf{J}_1] + [\mathbf{J}_3, \mathbf{J}_1]\mathbf{J}_3 = \mathbf{J}_2(-i\mathbf{J}_3) + (-i\mathbf{J}_3)\mathbf{J}_2 + \mathbf{J}_3(i\mathbf{J}_2) + i\mathbf{J}_2\mathbf{J}_3 = \mathbf{0}$

P4 **(4 pts)** Suppose that

$$\mathbf{J}^2\mathbf{A} = \alpha\mathbf{A} \qquad \mathbf{J}_3\mathbf{A} = \beta\mathbf{A} \tag{8}$$

where $\mathbf{A}$ is a operator and α, β are complex numbers. Prove that

$$\mathbf{J}^2(\mathbf{J}_\pm\mathbf{A}) = \alpha(\mathbf{J}_\pm\mathbf{A}) \qquad \mathbf{J}_3(\mathbf{J}_\pm\mathbf{A}) = (\beta \pm 1)(\mathbf{J}_\pm\mathbf{A}) \tag{9}$$

where you take all $+$'s or all $-$'s.

Solution: $\mathbf{J}^2\mathbf{J}_\pm = [\mathbf{J}^2, \mathbf{J}_\pm] + \mathbf{J}_\pm\mathbf{J}^2 = \mathbf{J}_\pm\mathbf{J}^2$ thus $\mathbf{J}^2(\mathbf{J}_\pm\mathbf{A}) = \mathbf{J}_\pm(\mathbf{J}^2\mathbf{A}) = \mathbf{J}_\pm(\alpha\mathbf{A}) = \alpha(\mathbf{J}_\pm\mathbf{A})$. Likewise, $\mathbf{J}_3\mathbf{J}_\pm = [\mathbf{J}_3, \mathbf{J}_\pm] + \mathbf{J}_\pm\mathbf{J}_3 = \pm\mathbf{J}_\pm + \mathbf{J}_\pm\mathbf{J}_3$ thus $\mathbf{J}_3(\mathbf{J}_\pm\mathbf{A}) = \pm\mathbf{J}_\pm\mathbf{A} + \mathbf{J}_\pm\mathbf{J}_3\mathbf{A} = (\beta \pm 1)(\mathbf{J}_\pm\mathbf{A})$.

P5 **(6 pts)** Suppose that we have an operator, $\mathbf{A}$ such that $[\mathbf{J}_3, \mathbf{A}] = k\mathbf{A}$ where k is a non-negative integer. For the sake of notation, write $\mathbf{T}_k = \mathbf{A}$, and define $\mathbf{T}_{q-1} = [\mathbf{J}_-, \mathbf{T}_q]$ for $q = k, k-1, k-2, \ldots$. Prove that $[\mathbf{J}_3, \mathbf{T}_q] = q\mathbf{T}_q$

Solution: Let us prove by induction that $[\mathbf{J}_3, \mathbf{T}_q] = q\mathbf{T}_q$ going from $q = k$ down. The base case $q = k$ is given. Now notice that by the Jacobi identity,

$$[\mathbf{J}_3, \mathbf{T}_{q-1}] = [\mathbf{J}_3, [\mathbf{J}_-, \mathbf{T}_q]] = -([\mathbf{J}_-, [\mathbf{T}_q, \mathbf{J}_3]] + [\mathbf{T}_q, [\mathbf{J}_3, \mathbf{J}_-]]) \tag{10}$$

Using $[\mathbf{J}_3, \mathbf{J}_-] = -\mathbf{J}_-$ and the inductive hypothesis gives us

$$= -([\mathbf{J}_-, -q\mathbf{T}_q] + [\mathbf{T}_q, -\mathbf{J}_-]) = (q-1)[\mathbf{J}_-, \mathbf{T}_q] = (q-1)\mathbf{T}_{q-1} \tag{11}$$

3 Counting with the Lie Product Formula (30 pts)

Operator Exponentials: Just as we can take a real number x, and then form a new real number, e^x, we can define the exponential of an operator using a similar formula:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \;\rightarrow\; \exp(\mathbf{A}) \equiv \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} \tag{12}$$

where $\mathbf{A}^0 = \mathbf{I}$ is the identity operator as defined in the introduction.

We know that for real numbers, $e^{x+y} = e^x e^y$. Suppose that instead of real numbers, we have operators, $\mathbf{X}$ and $\mathbf{Y}$. A natural question to ask is: what can we say about $\exp(\mathbf{X}+\mathbf{Y})$? While $\exp(\mathbf{X}+\mathbf{Y}) \neq \exp(\mathbf{X})\exp(\mathbf{Y})$ generally, the Lie product formula gives a way to represent $\exp(\mathbf{X}+\mathbf{Y})$ in terms of products of $\exp\left(\frac{\mathbf{X}}{n}\right)$ and $\exp\left(\frac{\mathbf{Y}}{n}\right)$:

$$\exp(\mathbf{X}+\mathbf{Y}) = \lim_{n\to\infty} \mathbf{M}_n \qquad \text{where} \qquad \mathbf{M}_n = \left(\exp\left(\frac{\mathbf{X}}{n}\right)\exp\left(\frac{\mathbf{Y}}{n}\right)\right)^n \tag{13}$$

If you are not familiar with limits, the intuitive idea is that if we take n to be very large, then the right side becomes the left side. We will not prove the lie product formula, but we will see that there are many counting problems hidden in the polynomial expansion of both sides. Let us consider the example of $n = 2$. In order to investigate this operator, we need to go back to the definition of the operator exponential and substitute it in to the previous equation to get:

$$\mathbf{M}_2 = \exp\left(\frac{\mathbf{X}}{2}\right)\exp\left(\frac{\mathbf{Y}}{2}\right)\exp\left(\frac{\mathbf{X}}{2}\right)\exp\left(\frac{\mathbf{Y}}{2}\right) \tag{14}$$

$$= \left(\mathbf{I} + \frac{\mathbf{X}}{2} + \frac{1}{2!}\left(\frac{\mathbf{X}}{2}\right)^2 + \ldots\right)\left(\mathbf{I} + \frac{\mathbf{Y}}{2} + \frac{1}{2!}\left(\frac{\mathbf{Y}}{2}\right)^2 + \ldots\right)\left(\mathbf{I} + \frac{\mathbf{X}}{2} + \frac{1}{2!}\left(\frac{\mathbf{X}}{2}\right)^2 + \ldots\right)\left(\mathbf{I} + \frac{\mathbf{Y}}{2} + \frac{1}{2!}\left(\frac{\mathbf{Y}}{2}\right)^2 + \ldots\right) \tag{15}$$

Recalling that $\mathbf{XY} \neq \mathbf{YX}$, we can expand out the above expression to get a polynomial function[1] of $\mathbf{X}$ and $\mathbf{Y}$.

P6 **(4 pts)** Find a, b, c, d, e, f, g if after expanding the expression for $\mathbf{M}_2$, we get:

$$\mathbf{M}_2 = a\mathbf{I} + b\mathbf{X} + c\mathbf{Y} + d\mathbf{X}^2 + e\mathbf{XY} + f\mathbf{YX} + g\mathbf{Y}^2 + \ldots \tag{16}$$

(No proof necessary). ***Solution:*** Expanding out straightforwardly gives $(a, b, c, d, e, f, g) = (1, 1, 1, 1/2, 3/4, 1/4, 1/2)$.

We call this finding the expansion of $\mathbf{M}_2$ to second order because we are finding the coefficients of all terms with degree less than equal to degree 2.

P7 **(7 pts)** Suppose that n is a positive integer. Find, with proof, the expansions of $\mathbf{M}_n$ and $\exp(\mathbf{X}+\mathbf{Y})$ to second order and show that their difference goes to zero if we let n go to infinity.

Solution: First, consider $\mathbf{M}_n$. There are $2n$ factors to expand out. Zeroth order is $1\mathbf{I}$. For first order, there are n ways to choose one $\frac{\mathbf{X}}{n}$ so we get $1\mathbf{X}$. Same with $1\mathbf{Y}$. Let us consider $\mathbf{X}^2$. There are two ways to do this. First, we can pick two factors of $\frac{\mathbf{X}}{n}$ for a coefficient of $\frac{\binom{n}{2}}{n^2}$. The other way is to choose one factor of $\mathbf{X}^2$. This can be done in n ways to get $\frac{n\frac{1}{2!}}{n^2}$. This gives a total coefficient of $\frac{1}{2}\mathbf{X}^2$. The same argument gives $\frac{1}{2}\mathbf{Y}^2$. For $\mathbf{XY}$, suppose that we choose $\mathbf{Y}$ from the jth factor where $j = 1, 2, \ldots n$. Then there are j ways to choose a $\mathbf{X}$. So the total coefficient is $\frac{\sum_{j=1}^{n} j}{n^2} = \frac{1}{2} + \frac{1}{2n}$. Finally

[1]Technically, polynomials have only a finite number of terms. The appropriate description of what we are doing is a formal power series expansion, but we will not be concerned with this distincition.

for $\mathbf{YX}$, we use the same argument, except we take $\mathbf{X}$ from the jth factor, and then there are $j-1$ choices for $\mathbf{Y}$. This gives us a coefficient of $\frac{\sum_{j=1}^{n-1} j}{n^2} = \frac{1}{2} - \frac{1}{2n}$.

Second, we can show that the expansion of $\exp(\mathbf{X}+\mathbf{Y})$ is $\mathbf{I}+\mathbf{X}+\mathbf{Y}+\frac{1}{2}\mathbf{X}^2+\frac{1}{2}\mathbf{XY}+\frac{1}{2}\mathbf{YX}+\frac{1}{2}\mathbf{Y}^2$.

Finally the difference is $\frac{\mathbf{XY}-\mathbf{YX}}{2n}$, which goes to zero as n goes to infinity.

P8 **(7 pts)** Find, with proof, the coefficient of $\mathbf{X}^{10}\mathbf{Y}^{10}$ in the expansion of $\mathbf{M}_2$.

Solution: Two cases. First, suppose that when we expand, we take at least one factor of $\mathbf{Y}$ from the first factor. Then no factors of $\mathbf{X}$ can be taken from the second factor with $\mathbf{X}$'s. Thus we can ignore the $\mathbf{X}$'s from the second factor. If we multiply $\exp(\mathbf{Y}/2)\exp(\mathbf{Y}/2)$, we get $\exp(\mathbf{Y})$. The coefficient of $\mathbf{Y}^{10}$ is $\frac{1}{10!}$. The coefficient from $\mathbf{X}^{10}$ is $\frac{1}{2^{10}10!}$. Second, We take all of our factors of $\mathbf{Y}$ from the second $\mathbf{Y}$ factor. Then all of the $\mathbf{X}$ factors come from the first two terms, and we use the same argument as before. We double counted a term $\frac{\mathbf{X}^{10}\mathbf{Y}^{10}}{10!2^{10}10!2^{10}}$ It follows that the answer is $\boxed{\frac{2^{11}-1}{2^{20}(10!)^2}}$.

P9 **(12 pts)** In the polynomial expansion of $\mathbf{M}_n$, find, with proof, the coefficient of $(\mathbf{XY})^k = \mathbf{XYXY}\ldots\mathbf{XY}$ where k is a positive integer.

Solution: It is sufficent to consider the expansion of $((\mathbf{I}+\mathbf{X})(\mathbf{I}+\mathbf{Y}))^n$ because there are no powers of $\mathbf{X}^2$ in the term that we want the coefficient of. In the end, we will multiply by $\frac{1}{n^{2k}}$ to account for $\mathbf{X}$ and $\mathbf{Y}$ being divided by n. When we expand out the expression above to get the desired coefficient, we choose which factor to pull out $\mathbf{X}$ and $\mathbf{Y}$. Let i_1 be the factor that the first $\mathbf{Y}$ comes from. i_2 is the number of factors after that factor to get the next factor of $\mathbf{Y}$, etc, until i_k. Given these placements of $\mathbf{Y}$'s, there are $i_1 \cdot \ldots \cdot l_k$ ways to choose the $\mathbf{X}$'s. Call $i_{k+1} = n - i_1 - \ldots - i_k \geq 0$ Thus the coefficient we want is

$$\sum_{\sum_{j=1}^{k+1} i_j = n} \prod_{j=1}^{k} i_j \tag{17}$$

where $i_1, \ldots, i_k \geq 1$ and $i_{k+1} \geq 0$. However, this is just the x^n coefficient of

$$(x + 2x^2 + 3x^3 + \ldots)^k (1 + x + x^2 + \ldots) = x^k (1-x)^{-(2k+1)} \tag{18}$$

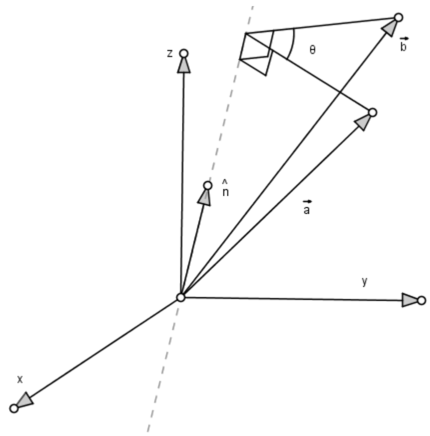
This is the $n-k$th coefficient of $(1-x)^{-(2k+1)}$, which is given by the binomal series:

$$(-1)^{n-k}\binom{-(2k+1)}{n-k-1} = (-1)^{n-k}\frac{(-2k-1)(-2k-2)\cdot\ldots\cdot(-2k-1-n+k+1)}{(n-k)!} = \binom{n+k}{2k} \tag{19}$$

It follows that the final answer for the coefficient is $\boxed{\frac{\binom{n+k}{2k}}{n^{2k}}}$

4 Rotations with Operator Exponentials (45 pts)

In this section, we will show that we can rotate vectors about axes using our noncommutative operators. First, let us explain the geometric problem: suppose that we are given the components of some vector $\vec{a}$ and we rotate that vector about the $\hat{n}$ axis an angle θ. Call the resulting vector $\vec{b}$. What are the components of $\vec{b}$?



Before we can proceed, we need two definitions:

Generator of Rotations Suppose that $\vec{\mathbf{J}} = (\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3)$ and $\vec{\mathbf{X}} = (\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ are two triples of operators. We say that $\vec{\mathbf{J}}$ generates vector rotations on $\vec{\mathbf{X}}$ if

1. $(\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3)$ forms an angular momentum algebra.
2. $[\mathbf{J}_i, \mathbf{X}_j] = i\sum_{k=1}^{3} \epsilon_{i,j,k}\mathbf{X}_k$. Note the i in front of the sum is $\sqrt{-1}$ and the subscript $_i$ is $1, 2, 3$.
3. $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are linearly independent.

In order to use the commutators, let us first associate the vectors $\vec{a} = (a_1, a_2, a_3)$ and so on with operators:

$$\begin{array}{ccc} \vec{a} = (a_1, a_2, a_3) & \leftrightarrow & \vec{a} \cdot \vec{\mathbf{X}} = a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + a_3\mathbf{X}_3 \\ \hat{n} = (n_1, n_2, n_3) & \leftrightarrow & \hat{n} \cdot \vec{\mathbf{J}} = n_1\mathbf{J}_1 + n_2\mathbf{J}_2 + n_3\mathbf{J}_3 \\ \vec{b} = (b_1, b_2, b_3) & \leftrightarrow & \vec{b} \cdot \vec{\mathbf{X}} = b_1\mathbf{X}_1 + b_2\mathbf{X}_2 + b_3\mathbf{X}_3 \end{array}$$

where $n_1^2 + n_2^2 + n_3^2 = 1$ and $\vec{\mathbf{J}}$ generates vector rotations on $\vec{\mathbf{X}}$. We will prove that $\vec{a}$ can be rotated about $\hat{n}$ an angle of θ using the following equation:

$$\exp(-i\theta\hat{n} \cdot \vec{\mathbf{J}})\vec{a} \cdot \vec{\mathbf{X}} \exp(i\theta\hat{n} \cdot \vec{\mathbf{J}}) = \vec{b} \cdot \vec{\mathbf{X}} \tag{20}$$

Let us begin with some problems. Be sure to use the definition of the exponential of an operator.

P10 **(7 pts)** Prove that $\exp((s + t)\mathbf{A}) = \exp(s\mathbf{A})\exp(t\mathbf{A})$.

Solution: We expand using the definition of the operator exponential

$$\exp(s\mathbf{A})\exp(t\mathbf{A}) = \sum_{m=0}^{\infty} s^m \frac{\mathbf{A}^m}{m!} \sum_{n=0}^{\infty} t^n \frac{\mathbf{A}^n}{n!} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\binom{m+n}{m}}{(m+n)!}\mathbf{A}^{m+n}s^m t^n \tag{21}$$

Rearranging to sum over specific powers of $\mathbf{A}$, we get

$$\sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{\binom{k}{l}}{k!}\mathbf{A}^k s^l t^{k-l} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}(s + t)^k = \exp((s + t)\mathbf{A}) \tag{22}$$

P11 The Hadamard Lemma:

(a) **(7 pts)** Suppose that $\mathbf{B}_0 = \mathbf{B}$ and that $\mathbf{B}_{n+1} = [\mathbf{A}, \mathbf{B}_n]$ for $n \geq 0$. Prove the following lemma for $N \geq 1$.

$$[\mathbf{A}^N, \mathbf{B}_0] = \sum_{j=1}^{N} \binom{N}{j} \mathbf{B}_j \mathbf{A}^{N-j} \tag{23}$$

Solution: Proceed by induction. $N = 1$ gives $[\mathbf{A}, \mathbf{B}_0] = \mathbf{B}_1$, which is clear. Now,

$$[\mathbf{A}^{N+1}, \mathbf{B}_0] = \mathbf{A}[\mathbf{A}^N, \mathbf{B}_0] + [\mathbf{A}, \mathbf{B}_0]\mathbf{A}^N \tag{24}$$

$$= \mathbf{A} \sum_{j=1}^{N} \binom{N}{j} \mathbf{B}_j \mathbf{A}^{N-j} + \mathbf{B}_1 \mathbf{A}^N = \sum_{j=1}^{N} \binom{N}{j} ([\mathbf{A}, \mathbf{B}_j] + \mathbf{B}_j \mathbf{A}) \mathbf{A}^{N-j} + \mathbf{B}_1 \mathbf{A}^N \tag{25}$$

$$= \sum_{j=1}^{N} \binom{N}{j} \mathbf{B}_{j+1} \mathbf{A}^{N-j} + \sum_{j=1}^{N} \binom{N}{j} \mathbf{B}_j \mathbf{A}^{N+1-j} + \mathbf{B}_1 \mathbf{A}^N \tag{26}$$

$$= \sum_{j=2}^{N+1} \binom{N}{j-1} \mathbf{B}_j \mathbf{A}^{N+1-j} + \sum_{j=1}^{N} \binom{N}{j} \mathbf{B}_j \mathbf{A}^{N+1-j} + \mathbf{B}_1 \mathbf{A}^N \tag{27}$$

$$= \mathbf{B}_{N+1} + \sum_{j=2}^{N} \left(\binom{N}{j-1} + \binom{N}{j} \right) \mathbf{B}_j \mathbf{A}^{N+1-j} + N\mathbf{B}_1 \mathbf{A}^N + \mathbf{B}_1 \mathbf{A}^N \tag{28}$$

Then using Pascal's identity and collecting the other terms back into the sum gives the desired result.

(b) **(4 pts)** Prove that[2]

$$[\exp(\mathbf{A}), \mathbf{B}_0] = \sum_{j=1}^{\infty} \frac{\mathbf{B}_j}{j!} \exp(\mathbf{A}) \tag{29}$$

Solution: Using the identity in the previous part,

$$[\exp(\mathbf{A}), \mathbf{B}_0] = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{j=1}^{N} \binom{N}{j} \mathbf{B}_j \mathbf{A}^{N-j} = \sum_{j=1}^{\infty} \sum_{N=j}^{\infty} \frac{1}{(N-j)!j!} \mathbf{B}_j \mathbf{A}^{N-j} \tag{30}$$

$$= \sum_{j=1}^{\infty} \mathbf{B}_j \sum_{M=0}^{\infty} \frac{1}{M!j!} \mathbf{A}^N = \sum_{j=1}^{\infty} \frac{\mathbf{B}_j}{j!} \exp(\mathbf{A}) \tag{31}$$

(c) **(3 pts)** Prove that

$$\exp(\mathbf{A})\mathbf{B}\exp(-\mathbf{A}) = \sum_{j=0}^{\infty} \frac{\mathbf{B}_j}{j!} \tag{32}$$

Solution: First, note that $\exp(\mathbf{A})\exp(-\mathbf{A}) = \exp(\mathbf{0}) = \mathbf{I}$ with $s = 1$, $t = -1$ from the identity in the start of this section. Using the previous equation, we multiply by both sides on the right by $\exp(-\mathbf{A})$ and then

$$[\exp(\mathbf{A}), \mathbf{B}_0] \exp(-\mathbf{A}) = \exp(\mathbf{A})\mathbf{B}\exp(-\mathbf{A}) - \mathbf{B}_0 \exp(\mathbf{A})\exp(-\mathbf{A}) \tag{33}$$

Moving $\mathbf{B}_0$ to the right gives us the result.

[2]Ignore issues of convergence.

P12 **(7 pts)** Suppose that $\vec{\mathbf{J}}$ generates rotations on $\vec{\mathbf{X}}$. Use the Hadamard Lemma to prove that

$$\exp(-i\theta\mathbf{J}_3)[x_1\mathbf{X}_1 + x_2\mathbf{X}_2 + x_3\mathbf{X}_3]\exp(i\theta\mathbf{J_3}) = a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + a_3\mathbf{X}_3 \tag{34}$$

where $(a_1, a_2, a_3) = (x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta, x_3)$. The following identities might be helpful:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \qquad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \tag{35}$$

Solution: This is a direct application of the previous lemma. Set $\mathbf{B}_0 = x_1\mathbf{X}_1 + x_2\mathbf{X}_2 + x_3\mathbf{X}_3$ and $\mathbf{A} = -i\theta\mathbf{J}_3$. Then

$\mathbf{B}_1 = [\mathbf{A}, \mathbf{B}_0] = -i\theta[\mathbf{J}_3, x_1\mathbf{X}_1 + x_2\mathbf{X}_2 + x_3\mathbf{X}_3] = \theta(x_1\mathbf{X}_2 - x_2\mathbf{X}_1)$.

$\mathbf{B}_2 = [\mathbf{A}, \mathbf{B}_1] = -i\theta^2[\mathbf{J}_3, -x_2\mathbf{X}_1 + x_1\mathbf{X}_2] = \theta^2((-x_2)\mathbf{X}_2 + (-x_1)\mathbf{X}_1)$,

$\mathbf{B}_3 = [\mathbf{A}, \mathbf{B}_2] = -\theta^3(x_1\mathbf{X}_2 - x_2\mathbf{X}_1)$

We notice that $\mathbf{B}_{n+2} = -\theta^2\mathbf{B}_n$ for $n \geq 1$ (if we ignore the $\mathbf{X}_3$ term in $\mathbf{B}_0$ (this can be proven inductively, for instance). Thus we split our sum into even and odd terms

$$\sum_{k=0,even}^{\infty} \frac{\mathbf{B}_k}{k!} = x_3\mathbf{X}_3 + (x_1\mathbf{X}_1 + x_2\mathbf{X}_2)\left(\frac{1}{0!} - \frac{\theta^2}{2!} + \frac{(-\theta^2)^2}{4!} + \dots\right) = x_3\mathbf{X}_3 + (x_1\mathbf{X}_1 + x_2\mathbf{X}_2)\cos\theta \tag{36}$$

Likewise,

$$\sum_{k=0,odd}^{\infty} \frac{\mathbf{B}_k}{k!} = (x_1\mathbf{X}_2 - x_2\mathbf{X}_1)\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right) = (x_1\mathbf{X}_2 - x_2\mathbf{X}_1)\sin\theta \tag{37}$$

Adding these two equations gives the result.

P13 (a) **(7 pts)** Starting with Eq. (20), prove that

$$(\vec{b} - \vec{a}) \cdot \vec{\mathbf{X}} = \theta(\hat{n} \times \vec{a}) \cdot \vec{\mathbf{X}} + \sum_{k=2}^{\infty} \theta^k \frac{\vec{a}_k}{k!} \cdot \vec{\mathbf{X}} \tag{38}$$

where $\vec{a}_k = \hat{n} \times \vec{a}_{k-1}$ and $\vec{a}_0 = \vec{a}$ and the dot product $\vec{v} \cdot \vec{\mathbf{X}}$ means $v_1\mathbf{X}_1 + v_2\mathbf{X}_2 + v_3\mathbf{X}_3$.

(b) **(6 pts)** Prove that

$$\left|\sum_{k=2}^{\infty} \theta^k \frac{\vec{a}_k}{k!}\right| \leq |\vec{a}|\theta^2 e^{|\theta|} \tag{39}$$

where $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ denotes the magnitude of the vector $\vec{v}$.

Solution:

(a) Using the Hadamard Lemma gives a series where

$$\vec{b} \cdot \vec{\mathbf{X}} = \vec{a} \cdot \vec{\mathbf{X}} + [-i\theta\hat{n} \cdot \vec{\mathbf{J}}, \vec{a} \cdot \vec{\mathbf{X}}] + \dots \tag{40}$$

It suffices to calculate the commutator. We can do this as follows:

$$-i\theta[\sum_{i=1}^{3} n_i\mathbf{J}_i, \sum_{j=1}^{3} a_j\mathbf{X}_j] = -i\theta\sum_{i=1}^{3}\sum_{j=1}^{3} n_i a_j[\mathbf{J}_i, \mathbf{X}_i] = \theta\sum_{i=1}^{3}\sum_{j=1}^{3} n_i a_j \sum_{k=1}^{3} \epsilon_{i,j,k}\mathbf{X}_k \tag{41}$$

Evaluating the sum and using the definition of $\epsilon_{i,j,k}$ gives us that $[-i\theta\hat{n} \cdot \vec{\mathbf{J}}, a\vec{\mathbf{X}}] = \theta\hat{n} \times \vec{a} \cdot \vec{\mathbf{X}}$. In other words, taking that commutator is the same as taking the cross product with $\theta\hat{n}$. Plugging this into the Hadamard lemma gives the result.

(b) Since $\hat{n}$ is a unit vector, we have

$$|\vec{a}_k| = |\hat{n} \times \vec{a}_{k-1}| \leq |\hat{n}||\vec{a}_{k-1}| \leq |\vec{a}_{k-1}| \tag{42}$$

Repeating this inductively gives $|\vec{a}_k| \leq |\vec{a}|$. Now we use the triangle inequality to prove that the LHS is less than or equal to

$$|\vec{a}|\theta^2 \sum_{k=2}^{\infty} \frac{|\theta|^{k-2}}{k!} \tag{43}$$

Finally, can finish the upper bound by doing:

$$\sum_{k=2}^{\infty} \frac{|\theta|^{k-2}}{k!} = \sum_{k=0}^{\infty} \frac{|\theta|^k}{(k+2)!} \leq \sum_{k=0}^{\infty} \frac{|\theta|^k}{k!} = e^{|\theta|} \tag{44}$$

P14 **(4 pts)** Give a geometric proof for why $\vec{b} - \vec{a} \approx \theta(\hat{n} \times \vec{a})$ when θ is much less than one. Clearly identify what approximation(s) you used in order to demonstrate this result.

Solution: Consider the diagram given in this section. Let O be the origin, A be the tip of $\vec{a}$, and B be the tip of $\vec{b}$. Furthermore, let the tip of the projection of $\vec{a}$ onto the $\hat{n}$ axis be point P. By the fact that we have a rotation, we know that OPA and OPB are right angles. Now, consider the difference $\vec{BA} = \vec{b} - \vec{a}$. We approximate the length AB as the circular arc AB using a circle centered at P with radius $PA = PB$. In this case, $AB \approx \theta \cdot AP$. However, $AP = AO \sin \angle AOP = |\vec{a} \times \vec{n}|$. Thus $AB = \theta|\vec{a} \times \vec{n}|$. Additionally, if θ is small, then the angle BAP is close to 90, so $\vec{b} - \vec{a}$ is approximately parallel to $\hat{n} \times \vec{a}$. Since we see that the magnitudes and directions approximately match, $\vec{b} - \vec{a} \approx \theta\hat{n} \times \vec{a}$.

5 Spherical Tensors (40 pts)

This section was not included in the original contest

While we have seen how a simple set of commutation relations capture the essence of rotations of three dimensional vectors, there is vast and rich structure associated with rotations of more general objects in 3 dimensions. Suppose that

1. $\vec{\mathbf{A}}$ generates vector rotations on $\vec{\mathbf{X}}$ (see the previous section).

2. $\vec{\mathbf{B}}$ generates vector rotations on $\vec{\mathbf{Y}}$.

3. These two rotations are independent in the sense that for all i, j in $\{1, 2, 3\}$,

$$[\mathbf{A}_i, \mathbf{B}_j] = \mathbf{0} \qquad [\mathbf{A}_i, \mathbf{Y}_j] = \mathbf{0} \qquad [\mathbf{X}_i, \mathbf{B}_j] = \mathbf{0} \qquad [\mathbf{X}_i, \mathbf{Y}_j] = \mathbf{0} \tag{45}$$

Define the total angular momentum $\mathbf{J}_i = \mathbf{A}_i + \mathbf{B}_i$ for $i = 1, 2, 3$. We will consider how $\mathbf{J}$ generates (not necessarily vector) rotations on the nine objects $\mathbf{T}_{i,j} = \mathbf{X}_i\mathbf{Y}_j$ where $i, j \in \{1, 2, 3\}$. To study this question, we will look at $[\mathbf{J}_i, \mathbf{T}_{j,k}]$. In a sense, we will recombine these nine objects and then divide them into three types. One type that doesn't change at all under a rotation, one that behaves like a vector under rotation, and one with a more general behavior.

P15 Suppose that $\mathbf{J}_i = \mathbf{A}_i + \mathbf{B}_i$ for $i = 1, 2, 3$.

(a) **(3 pts)** Prove that $[\mathbf{J}_i, \mathbf{J}_j] = i\sum_{k=1}^{3} \epsilon_{i,j,k}\mathbf{J}_k$.

(b) **(3 pts)** Prove that $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$ are linearly independent using the fact that the $\mathbf{A}$'s and the $\mathbf{B}$'s are linearly independent (see the definition of linearly independent in section 2).

Solution:

(a) Since $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are angular momentum algebras and the given commutation relations,

$$[\mathbf{J}_i, \mathbf{J}_j] = [\mathbf{A}_i + \mathbf{B}_i, \mathbf{A}_j + \mathbf{B}_j] = [\mathbf{A}_i, \mathbf{A}_j] + [\mathbf{A}_i, \mathbf{B}_j] + [\mathbf{B}_i, \mathbf{A}_j] + [\mathbf{B}_i, \mathbf{B}_j] \tag{46}$$

$$i\sum_{k=1}^{3} \epsilon_{i,j,k}\mathbf{A}_k + \mathbf{0} + \mathbf{0} + i\sum_{k=1}^{3} \epsilon_{i,j,k}\mathbf{B}_k = i\sum_{k=1}^{3} \epsilon_{i,j,k}\mathbf{J}_k \tag{47}$$

(b) Suppose that

$$\mathbf{0} = c_1(\mathbf{A}_1 + \mathbf{B}_1) + c_2(\mathbf{A}_2 + \mathbf{B}_2) + c_2(\mathbf{A}_2 + \mathbf{B}_2) \tag{48}$$

Taking the commutator of both sides with $\mathbf{A}_1$ gives us $c_2\mathbf{A_3} - c_3\mathbf{A_2} = \mathbf{0}$. Therefore by the linear independence of the $\mathbf{A}$'s, we get $c_2 = c_3 = 0$. Repeating with say $\mathbf{A}_2$ gives $c_1 = 0$ as well.

Transforming like a rank k Spherical Tensor: A collection of $2k+1$ operators $\mathbf{T}_{-k}$,$\mathbf{T}_{-k+1}$,...,$\mathbf{T}_k$ is said to transform like a rank k spherical tensor under $\vec{\mathbf{J}}$ if the following are true:

1. $(\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3)$ forms an angular momentum algebra.
2. $[\mathbf{J}_3, \mathbf{T}_q] = q\mathbf{T}_q$ for $q = -k, -k+1, ..., k$.
3. $[\mathbf{J}_\pm, \mathbf{T}_q] = \sqrt{(k \mp q)(k \pm q + 1)}\mathbf{T}_{q\pm 1}$ for $q = -k, -k+1, ..., k$ (take all upper signs or all lower signs).[3]
4. $\mathbf{T}_{-k}$,$\mathbf{T}_{-k+1}$,...,$\mathbf{T}_k$ are linearly independent.

where $\mathbf{J}_\pm = \mathbf{J}_1 \pm i\mathbf{J}_2$.

P16 **(4 pts)** Prove that

$$\mathbf{T}_0 = \vec{\mathbf{X}} \cdot \vec{\mathbf{Y}} \equiv \mathbf{X}_1\mathbf{Y}_1 + \mathbf{X}_2\mathbf{Y}_2 + \mathbf{X}_3\mathbf{Y}_3 \tag{49}$$

transforms like a rank 0 tensor under $\vec{\mathbf{J}}$. In other words, prove that $[\mathbf{J}_\alpha, \mathbf{T}_0] = \mathbf{0}$ where α is replaced with $3, +, -$.

Solution: Observe that it is equivalent to prove this with $\alpha = 1, 2, 3$. Then compute:

$$\left[\mathbf{J}_i, \sum_{j=1}^{3}\mathbf{X}_j\mathbf{Y}_j\right] = \sum_{j=1}^{3}[\mathbf{J}_i, \mathbf{X}_j\mathbf{Y}_j] = \sum_{j=1}^{3}\mathbf{X}_j\left[\mathbf{J}_i, \mathbf{Y}_j\right] + \left[\mathbf{J}_i, \mathbf{X}_j\right]\mathbf{Y}_j = i\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{i,j,k}(\mathbf{X}_j\mathbf{Y}_k + \mathbf{X}_k\mathbf{Y}_j) \tag{50}$$

Now we note that $\epsilon_{i,j,k} = -\epsilon_{i,k,j}$, split up the sums, and interchanging the dummy variables j, k in the second sum.

$$= i\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{i,j,k}\mathbf{X}_j\mathbf{Y}_k + i\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{i,j,k}\mathbf{X}_k\mathbf{Y}_j = i\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{i,j,k}\mathbf{X}_j\mathbf{Y}_k + i\sum_{j=1}^{3}\sum_{k=1}^{3} -\epsilon_{i,k,j}\mathbf{X}_k\mathbf{Y}_j \tag{51}$$

$$= i\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{i,j,k}\mathbf{X}_j\mathbf{Y}_k + i\sum_{j=1}^{3}\sum_{k=1}^{3} -\epsilon_{i,j,k}\mathbf{X}_j\mathbf{Y}_k = \mathbf{0} \tag{52}$$

Span: Let $\mathbf{M}_1, \mathbf{M}_2, .., \mathbf{M}_k$ be operators. Then the span of $\mathbf{M}_1, \ldots, \mathbf{M}_k$ is the set of all elements of the form $c_1\mathbf{M}_1 + c_2\mathbf{M}_2 + \ldots + c_k\mathbf{M}_k$ where $c_1, \ldots, c_k$ are complex numbers.

P17 (a) **(4 pts)** Find three operators $\mathbf{T}_{-1}, \mathbf{T}_0, \mathbf{T}_1$ in Span$(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ such that both of the following are true:

i. $\mathbf{T}_q$ where $q = -1, 0, 1$ transforms like a spherical tensor of rank 1

[3] Note for $+$ and $q = k$, we have $[\mathbf{J}_+, \mathbf{T}_k] = \sqrt{(k-k)(k+k+1)}\mathbf{T}_{k+1} = \mathbf{0}$ so we don't have to worry about $\mathbf{T}_{k+1}$ technically not being defined and we just say that $[\mathbf{J}_+, \mathbf{T}_k] = \mathbf{0}$. Similarly with $-$ and $q = -k$.

ii. If $\mathbf{T}_0 = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + c_3\mathbf{X}_3$, then $c_3 \geq 0$ and $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$.

(b) **(10 pts)** Let V be the span of the nine operators $\mathbf{X}_i\mathbf{Y}_j$ where $i, j \in \{1, 2, 3\}$. Find three operators in V that transform like a spherical tensor of rank 1. Hint: Prove that if $\mathbf{Z}_j = \sum_{l=1}^{3}\sum_{m=1}^{3} \epsilon_{j,l,m}\mathbf{X}_l\mathbf{Y}_m$, then $\mathbf{J}$ generates vector rotations on $\vec{\mathbf{Z}}$.

Solution:

(a) Since $\mathbf{T}_0$ is in the span the $\mathbf{X}_i$'s, we can write it as in (b). Plugging that into the first property of spherical tensors for $k = 1$, $q = 0$, we get $0\mathbf{T}_0 = [\mathbf{J}_3, \mathbf{T}_0] = c_1\mathbf{X}_2 - c_2\mathbf{X}_1$. It follows that $c_1 = c_2 = 0$ by the linear independence of the $\mathbf{X}$'s. Thus we have $c_3 = 1$. Now we apply the second condition with $k = 1$, $q = 0$ to get

$$[\mathbf{J}_\pm, \mathbf{T}_0] = \sqrt{(1 \mp 0)(1 \pm 0 + 1)}\mathbf{T}_{\pm 1} = [\mathbf{J}_1, \mathbf{X}_3] \pm i[\mathbf{J}_2, \mathbf{X}_3] \tag{53}$$

It follows that

$$\mathbf{T}_{\pm 1} = \mp\frac{\mathbf{X}_1 \pm i\mathbf{X}_2}{\sqrt{2}} \qquad \mathbf{T}_0 = \mathbf{X}_3 \tag{54}$$

To finish the question, we just need to verify the first and second property for $q = \pm 1$, $k = 1$. Lastly, we need to verify linear independence.

$$c_1(-1)\frac{\mathbf{X}_1 + i\mathbf{X}_2}{\sqrt{2}} + c_0\mathbf{X}_3 + c_{-1}\frac{\mathbf{X}_1 - i\mathbf{X}_2}{\sqrt{2}} = \mathbf{0} \tag{55}$$

It follows that $c_0 = 0$, $c_1 + c_{-1} = 0$ and $c_1 - c_{-1} = 0$ so $c_1 = c_0 = c_{-1} = 0$.

(b) If we prove the hint, then we are done by the first part because we can recombine the comonents from the vector to get the rank 1 tensor. Let us move on to prove the hint. For convenience, we employ Einstein's summation convention where if there are two instances of an index, we implicitly assume that there is a sum there.

$$[\mathbf{J}_i, \mathbf{Z}_j] = [\mathbf{J}_i, \epsilon_{j,k,l}\mathbf{X}_k\mathbf{Y}_l] = \epsilon_{j,k,l}(\mathbf{X}_k[\mathbf{J}_i, \mathbf{Y}_l] + [\mathbf{J}_i, \mathbf{X}_k]\mathbf{Y}_l) \tag{56}$$

$$= i\epsilon_{j,k,l}(\mathbf{X}_k\epsilon_{i,l,m}\mathbf{Y}_m + \epsilon_{i,k,m}\mathbf{X}_m\mathbf{Y}_l) \tag{57}$$

One can verify that

$$\epsilon_{j,k,l}\epsilon_{i,l,m} = \delta_{i,k}\delta_{m,j} - \delta_{i,j}\delta_{k,m} \qquad \epsilon_{j,k,l}\epsilon_{i,k,m} = \delta_{i,j}\delta_{l,m} - \delta_{j,m}\delta_{l,i} \tag{58}$$

From which it follows that

$$= i(\mathbf{X}_i\mathbf{Y}_j - \delta_{i,j}\mathbf{X}_k\mathbf{Y}_k + \delta_{i,j}\mathbf{X}_m\mathbf{Y}_m - \mathbf{X}_j\mathbf{Y}_i) = i(\mathbf{X}_i\mathbf{Y}_j - \mathbf{X}_j\mathbf{Y}_i) \tag{59}$$

Correspondingly, we compute

$$\epsilon_{i,j,k}\mathbf{Z}_k = \epsilon_{i,j,k}\epsilon_{k,l,m}\mathbf{X}_l\mathbf{Y}_m = (\delta_{i,l}\delta_{j,m} - \delta_{i,m}\delta_{j,l})\mathbf{X}_l\mathbf{Y}_m = \mathbf{X}_i\mathbf{Y}_j - \mathbf{X}_j\mathbf{Y}_i \tag{60}$$

Thus it follows that

$$[\mathbf{J}_i, \mathbf{Z}_j] = i\epsilon_{i,j,k}\mathbf{Z}_k \tag{61}$$

P18 (a) **(10 pts)** Suppose that we have an operator, $\mathbf{A}$ such that $[\mathbf{J}_3, \mathbf{A}] = k\mathbf{A}$ and $[\mathbf{J}_+, \mathbf{A}] = \mathbf{0}$ where $2k$ is a non-negative integer. For the sake of notation, write $\mathbf{T}_k = \mathbf{A}$, and define $\mathbf{T}_{q-1} = \frac{1}{\sqrt{(k+q)(k-q+1)}}[\mathbf{J}_-, \mathbf{T}_q]$ for $q = k, k-1, ..., -k$. Assume that $\mathbf{T}_q \neq \mathbf{0}$ for $q = k, k-1, ..., -k$, but $\mathbf{T}_{-k-1} = \mathbf{0}$. Prove that the $2k+1$ matrices $\mathbf{T}_q$ for $q = -k, ..., k$ form a spherical tensor of rank k. Hint: Use the Jacobi Identity to calculate commutators like $[\mathbf{J}_3, [\mathbf{J}_-, \mathbf{T}_q]]$.

(b) **(6 pts)** Let V be the span of the nine operators $\mathbf{X}_i\mathbf{Y}_j$ where $i, j \in \{1, 2, 3\}$. Find five operators in V that transform like a spherical tensor of rank 2.

Solution:

(a) The part of the relations with the $\mathbf{J}_-$ are satisfied by construction. Now lets look at the $\mathbf{J}_3$ identities. Let us prove by induction that $[\mathbf{J}_3, \mathbf{T}_q] = q\mathbf{T}_q$ going from $q = k$ down to $q = -k$. The base case $q = k$ is given. Now notice that by the Jacobi identity, for $\alpha = \frac{1}{\sqrt{(k+q)(k-q+1)}}$

$$[\mathbf{J}_3, \mathbf{T}_{q-1}] = [\mathbf{J}_3, \alpha[\mathbf{J}_-, \mathbf{T}_q]] = -\alpha([\mathbf{J}_-, [\mathbf{T}_q, \mathbf{J}_3]] + [\mathbf{T}_q, [\mathbf{J}_3, \mathbf{J}_-]]) \tag{62}$$

Using $[\mathbf{J}_3, \mathbf{J}_-] = -\mathbf{J}_-$ and the inductive hypothesis gives us

$$= -\alpha([\mathbf{J}_-, -q\mathbf{T}_q] + [\mathbf{T}_q, -\mathbf{J}_-]) = (q-1)\alpha[\mathbf{J}_-, \mathbf{T}_q] = (q-1)\mathbf{T}_{q-1} \tag{63}$$

Now lets prove the $\mathbf{J}_+$ type identities. We will prove by induction that $[\mathbf{J}_+, \mathbf{T}_q] = \sqrt{(k-q)(k+q+1)}\mathbf{T}_{q+1}$ for $q = k, k-1, ..., -k$. The base case $q = k$ is given. Following the strategy from before, for $\alpha = \frac{1}{\sqrt{(k+q)(k-q+1)}}$

$$[\mathbf{J}_+, \mathbf{T}_{q-1}] = [\mathbf{J}_+, \alpha[\mathbf{J}_-, \mathbf{T}_q]] = -\alpha([\mathbf{J}_-, [\mathbf{T}_q, \mathbf{J}_+]] + [\mathbf{T}_q, [\mathbf{J}_+, \mathbf{J}_-]]) \tag{64}$$

Using the inductive hypothesis and what we we proved previously, we get

$$-[\mathbf{J}_-, [\mathbf{T}_q, \mathbf{J}_+]] = [\mathbf{J}_-, \sqrt{(k-q)(k+q+1)}\mathbf{T}_{q+1}]\sqrt{(k-q)(k+q+1)}\sqrt{(k+q+1)(k-(q+1)+1)}\mathbf{T}_q \tag{65}$$

Using $[\mathbf{J}_+, \mathbf{J}_-] = 2\mathbf{J}_3$, $-[\mathbf{T}_q, [\mathbf{J}_+, \mathbf{J}_-]] = -[\mathbf{T}_q, 2\mathbf{J}_3] = 2q\mathbf{T}_q$. Combining these relations,

$$[\mathbf{J}_+, \mathbf{T}_{q-1}] = \alpha((k-q)(k+q+1) + 2q)\mathbf{T}_q) \tag{66}$$

Then noting that $\alpha(k-q)(k+q+1) + 2q = \alpha(k+q)(k-q+1) = \sqrt{(k+q)(k-q-1)} = \sqrt{(k-(q-1))(k+(q-1)+1)}$, we get the result.

Lastly, we must prove linear independence. Suppose that $\sum_{q=-k}^{k} c_q\mathbf{T}_q = \mathbf{0}$. Suppose that Q is the largest integer such that $c_Q \neq 0$. Now take commutators of both sides with $\mathbf{J}_-$ $Q+k$ times. This gives us $c_Q(constants)\mathbf{T}_{-k} = \mathbf{0}$. But it is assumed that $\mathbf{T}_{-k} \neq \mathbf{0}$, so $c_Q = 0$, a contradiction. Thus all $c_q = 0$.

(b) Use the construction in the previous problem to take $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ to a spherical tensor of rank 1: $\mathbf{X}'_q$ for $q = -1, 0, 1$. Do the same with $\vec{\mathbf{Y}}$. Now we use the first part of this question, and note that

$$[\mathbf{J}_3, \mathbf{X}'_1\mathbf{Y}'_1] = [\mathbf{A}_3, \mathbf{X}'_1]\mathbf{Y}'_1 + \mathbf{X}'_1[\mathbf{A}_3, \mathbf{Y}'_1] + [\mathbf{B}_3, \mathbf{X}'_1]\mathbf{Y}'_1 + \mathbf{X}'_1[\mathbf{B}_3, \mathbf{Y}'_1] = 2\mathbf{X}'_1\mathbf{Y}'_1 \tag{67}$$

Thus our $\mathbf{A} = \mathbf{X}'_1\mathbf{Y}'_1$. To finish, we just need to repeatedly apply $\mathbf{J}_-$. Then we get

$$\mathbf{T}_1 = \frac{1}{\sqrt{2}}(\mathbf{X}'_1\mathbf{Y}'_0 + \mathbf{X}'_0\mathbf{Y}'_1) \qquad \mathbf{T}_0 = \frac{1}{\sqrt{6}}(\mathbf{X}'_1\mathbf{Y}'_{-1} + 2\mathbf{X}'_0\mathbf{Y}'_0 + \mathbf{X}'_{-1}\mathbf{Y}'_1) \tag{68}$$

$$\mathbf{T}_{-1} = \frac{1}{\sqrt{2}}(\mathbf{X}'_{-1}\mathbf{Y}'_0 + \mathbf{X}'_0\mathbf{Y}'_{-1}) \qquad \mathbf{T}_{-2} = \mathbf{X}'_{-1}\mathbf{Y}'_{-1} \tag{69}$$

Then the final answers follow from plugging in the following relations (and with $\mathbf{X} \to \mathbf{Y}$).

$$\mathbf{X}'_{\pm 1} = \mp\frac{\mathbf{X}_1 \pm i\mathbf{X}_2}{\sqrt{2}} \qquad \mathbf{X}_0 = \mathbf{X}_3 \tag{70}$$

Operator Space: We will call V a operator space if the following are true:

(a) V is nonempty set of operators.

(b) If $\mathbf{A} \in V$ implies that $c\mathbf{A} \in V$ for all complex numbers, c.

(c) If $\mathbf{A} \in V$ and $\mathbf{B} \in V$, then $(\mathbf{A}+\mathbf{B}) \in V$.

P19 **(10 pts)** Suppose that the collection of $2k+1$ operators $\mathbf{T}_q$ operators for $q = -k, -k+1, ..., k$ transform like a rank k spherical tensor. Let $V = \text{Span}\,(\mathbf{T}_{-k}, \mathbf{T}_{-k+1}, ..., \mathbf{T}_k)$. Let $f_i(\mathbf{A}) = [\mathbf{J}_i, \mathbf{A}]$ for $i = 1, 2, 3$. Suppose that $S \subset V$ is an operator space such that $S \neq \{\mathbf{0}\}$ and $\mathbf{A} \in S$ implies $[\mathbf{J}_i, \mathbf{A}] \in S$. Prove that $S = V$.

6 Calculating operator Exponentials (XX pts)

This section was not included in the original contest. Note that an explicit example of operators are n by n matrices. This section has one do some computations.

Hint: Begin by computing $\mathbf{A}^2 = \mathbf{A}\mathbf{A}$, $\mathbf{A}^3 = \mathbf{A}\mathbf{A}\mathbf{A}$, etc.

1. **(2 pts)** Find $\exp(\mathbf{A})$ where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{71}$$

Note: if you found that $\mathbf{A}^2 = \mathbf{A}$, you are not multiplying the operators correctly and you should see the appendix to make sure that you understand how to multiply operators.

Solution: Computing powers of A, we see that $\mathbf{A}^4 = \mathbf{0}$, so the answer just consists of the first three terms of the summation, which give us

$$\exp(\mathbf{A}) = \begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{72}$$

2. **(4 pts)** Find $\exp(\mathbf{B})$ where

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{73}$$

Solution: We can show by induction that

$$\mathbf{B}^n = \begin{bmatrix} 1 & 0 & 0 & n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{74}$$

Now we sum. The diagonal entries, we get $\sum_{k=0}^{\infty} \frac{1}{k!} = e$. In the top right, we get $\sum_{k=0}^{\infty} \frac{k}{k!} = \sum_{k=1}^{\infty} \frac{k}{k!} = e$. Thus the answer is

$$\exp(\mathbf{B}) = \begin{bmatrix} e & 0 & 0 & e \\ 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & e \end{bmatrix} \tag{75}$$

3. **(7 pts)** Find $\exp(\mathbf{C})$ where

$$\mathbf{C} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{76}$$

Solution: One can observe a pattern and use induction, however, here is a solution that offers a bit more insight. Observe that $\mathbf{C} = \mathbf{I} + \mathbf{A}$ where $\mathbf{A}^4 = 0$. Thus

$$\mathbf{C}^n = (\mathbf{I} + \mathbf{A})^n = \sum_{j=0}^{n} \binom{n}{j} \mathbf{A}^j = \sum_{j=0}^{3} \binom{n}{j} \mathbf{A}^j \tag{77}$$

where $\binom{n}{j} = 0$ if $j > n$. Now we just sum to get

$$\exp(\mathbf{C}) = \sum_{n=0}^{\infty} \sum_{j=0}^{3} \binom{n}{j} \frac{\mathbf{A}^j}{n!} = \sum_{j=0}^{3} \mathbf{A}^j \sum_{n=0}^{\infty} \frac{\binom{n}{j}}{n!} \tag{78}$$

Now,

$$\sum_{n=0}^{\infty} \frac{\binom{n}{j}}{n!} = \sum_{n=j}^{\infty} \frac{1}{j! \cdot (n-j)!} = \frac{e}{j!} \tag{79}$$

It follows that the answer is

$$\exp(\mathbf{C}) = \begin{bmatrix} e & e & e/2 & e/6 \\ 0 & e & e & e/2 \\ 0 & 0 & e & e \\ 0 & 0 & 0 & e \end{bmatrix} \tag{80}$$

7 The SU(2) Angular Momentum Algebra (XX pts)

This section also did not appear in the actual contest. Here is a specific realization of an angular momentum algebra in terms of matrices.

1. So far, we have considered angular momentum algebras to be abstract objects that we can manipulate without refering to particular operators that satisfy the relations in Eq. (**??**). Now we will consider an example. Define the operators

$$\mathbf{A}_1 = \frac{1}{2} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{A}_2 = \frac{1}{2} \cdot \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \mathbf{A}_3 = \frac{1}{2} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{81}$$

2. **(3 pts)** Show that the operators $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ satisfy Eq. (**??**).

3. **(3 pts)** Find all 2 by 1 operators, $\mathbf{v}$, with corresponding complex numbers λ_1, λ_2 such that $\mathbf{A}^2\mathbf{v} = \lambda_1 \mathbf{v}$ and $\mathbf{A}_3\mathbf{v} = \lambda_2 \mathbf{v}$.

4. Write $\exp(\mathbf{A})\mathbf{B}\exp(-\mathbf{A})$ as $\sum_{i=1}^{3} c_i \mathbf{A}_i$ for the following cases:
   (a) **(2 pts)** $\mathbf{A} = ix\mathbf{A}_3$, $B = \mathbf{A}_1$.
   (b) **(4 pts)** $\mathbf{A} = ix\mathbf{A}_1$, $B = \mathbf{A}_2$.

8 The Eigenvalues of J^2 and J_3 (24 pts)

This section was not included in the original contest. The focus of this section is that the eigenvalues of the operators of an angular momentum algebra can only take on discrete values!

1. *Matrix eigenvalues* Suppose that we have a matrix, M. A column vector, v, is called an eigenvector of M if and only if

$$v \neq 0 \tag{82}$$

and there exists a complex number, λ, which is called the eigenvalue of v with respect to M, such that

$$Mv = \lambda v \tag{83}$$

2. Introduction: Since J^2 and J_3 commute, it can be shown that they have a simultaneous eigenvector. Let $v_{a,b}$ denote a simultaneous eigenvector of J^2 and J_3 that is normalized. In particular, $v_{a,b} \neq 0$ is a column vector that satisfies the following:

$$(v_{a,b})^\dagger v_{a,b} = [1] \tag{84}$$

$$J^2 v_{a,b} = a\, v_{a,b} \qquad J_3 v_{a,b} = b\, v_{a,b} \tag{85}$$

where a, b are complex numbers.

Furthermore, we will make an additional assumption:

Suppose that for some complex numbers, a, b, there exists is a nonzero column vector, w such that $J^2 w = a\, w$ and $J_3 w = b\, w$. We will assume that then there exists a complex number, c, such that $w = c\, v_{a,b}$ where $v_{a,b}$ is some column vector such that Eqs. (84) and (85) are true.

3. **(4 pts)** Consider $w_\pm = J_\pm v_{a,b}$. Show that either (1) $w_\pm = 0$, or (2) $w_\pm$ is an eigenvector of J^2 and J_3 with eigenvalues of $a, b \pm 1$, respectively.

4. **(4 pts)** In the first case, argue that $a = b^2 \pm b$.

5. **(4 pts)** In the second case, argue that $J_\pm v_{a,b}$ can be written as $c^\pm_{a,b}\, v_{a,b\pm 1}$ where $c^\pm_{a,b}$ is a complex number that satisfies $|c^\pm_{a,b}|^2 = a - b^2 \mp b$.

6. **(2 pts)** Suppose that $v_{a,b}$ satisfies Eqs. (84) and Eq. (85). Prove that $|b| \leq \sqrt{a + \frac{1}{4}} + \frac{1}{2}$.

7. **(10 pts)** Begin with the existence of $v_{a,b}$ as in Eqs. (84) and Eq. (85). Then argue that there must be a b_{max} and b_{min} such that the following are true:

$$J_+ v_{a,b_{max}} = J_- v_{a,b_{min}} = 0 \tag{86}$$

$$b_{max} = b_{min} + 2j \tag{87}$$

where $2j$ is a non-negative integer. Conclude that the following are true:

$$b_{max} = -b_{min} = j \tag{88}$$

$$b \in \{-j, -j+1, \ldots, j-1, j\} \tag{89}$$

$$a = j(j+1) \tag{90}$$