1. A rectangle with sides a and b has an area of 24 and a diagonal of length 11. Find the perimeter of this rectangle.
Answer: 13
Solution: We have $ab = 24$ and $a^2 + b^2 = 121$. Thus, $a + b = \sqrt{169} = 13$.

2. Two rays start from a common point and have an angle of 60 degrees. Circle C is drawn with radius 42 such that it is tangent to the two rays. Find the radius of the circle that has radius smaller than circle C and is also tangent to C and the two rays.
Answer: 21
Solution: Let the intersection of the rays be A. Also let O be the center of circle C and P be the center of the smaller circle. Draw line OP and extend it such that it intersects A. We know these points are collinear because O and P are equidistant from the rays (verified by dropping perpendicular lines from O and P to the rays). Also, line OA bisects the angle between the rays. Now draw a line from O to a ray but is perpendicular to the ray. Call this point M. We know that $\angle OAM$ is 30 degrees, and that $\angle AMO$ is right (perpendicular). Thus $\angle APM$ is 30 degrees. Now draw a perpendicular line from P to OM, and call this point N. Let the radius we want to find be r. We know that $OP = r + 42$ and that $ON = OM - NM = 42 - r$. Now we can equate $cos(60) = \frac{ON}{OP} = \frac{r+42}{r-42} = \frac{1}{2}$. Solving for r we get 21.

3. Given a regular tetrahedron $ABCD$ with center O, find $\sin\angle AOB$.
Answer: $\frac{2\sqrt{2}}{3}$
Solution: Assume the edge length is 1. Let E be the center of triangle ABC. The height of triangle ABC is $\frac{\sqrt{3}}{2}$,so AE=$\frac{2}{3}\frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}$. Triangle AED is a right triangle, so the tetrahedron's height DE=$\sqrt{\frac{2}{3}}$. Thus, DO is $\frac{3}{4}\sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{4}$. Thus AOB is an isosceles triangle with base length 1 and leg length $\frac{\sqrt{6}}{4}$. Letting θ be angle AOB, and using the Law of Cosines:

$$1 = \frac{3}{8} + \frac{3}{8} - \frac{3}{4}cos(\theta)$$

, so $\theta = \cos^{-1}\left(-\frac{1}{3}\right)$. Then, $\sin\theta = \boxed{\frac{2\sqrt{2}}{3}}$.

4. Two cubes A and B have different side lengths, such that the volume of cube A is numerically equal to the surface area of cube B. If the surface area of cube A is numerically equal to six times the side length of cube B, what is the ratio of the surface area of cube A to the volume of cube B?
Answer: $\frac{1}{1296}$
Solution: If cube A has side length a and cube B has side length b, then $a^3 = 6b^2$ and $6a^2 = 6b$, so $\frac{1}{a} = 6$ or $a = \frac{1}{6}$ and $b = \frac{1}{36}$. Then, $\frac{6a^2}{b^3} = \frac{1}{b^2} = \boxed{\frac{1}{1296}}$.

5. Points A and B are fixed points in the plane such that $AB = 1$. Find the area of the region consisting of all points P such that $\angle APB > 120°$.
Answer: $\frac{2\pi}{9} - \frac{\sqrt{3}}{6}$
Solution: Pick a point Q such that $\angle AQB = 120$. Let O be the center of the circumcircle of $\triangle AQB$ and let M be the midpoint of AB. Note that any point within the region of the circle formed by the segment AB and the minor arc AB will satisfy the requirements for P. The area of this region is equal to the area of the sector formed by AB and the triangle AOB. Since $\angle AOB = 120°, \angle MOB = 60°$. We know $\angle BMO = 90°$, so since $AM = \frac{1}{2}$, we have $OM = \frac{1}{2\sqrt{3}}$ and $OB = \frac{1}{\sqrt{3}}$. This gives $[\triangle AOB] = \frac{1}{2} \cdot \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{12}$ and the area of the sector is $\frac{1}{3}\pi\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{9}$. Then the area of the region is $\left(\frac{\pi}{9} - \frac{\sqrt{3}}{6}\right)$. Note that we can construct two such regions, one on each side of segment AB. Then the answer is twice the area of one such region, giving us $2\left(\frac{\pi}{9} - \frac{\sqrt{3}}{6}\right) = \frac{2\pi}{9} - \frac{\sqrt{3}}{6}$.

6. Let $ABCD$ be a cyclic quadrilateral where $AB = 4$, $BC = 11$, $CD = 8$, and $DA = 5$. If BC and DA intersect at X, find the area of $\triangle XAB$.
Answer: $6\sqrt{5}$
Solution: Denote the lengths of XA, XB by x, y, respectively. Since $ABCD$ is cyclic, the triangles XAB and XCD are similar (AA) and we get

$$\frac{y+11}{x} = \frac{8}{4} = \frac{x+5}{y},$$

so $y + 11 = 2x$ and $x + 5 = 2y$. Solving gives $x = 7$, $y = 9$. Recalling Heron's formula we obtain the answer $[XAB] = \sqrt{10 \cdot 6 \cdot 3 \cdot 1} = 6\sqrt{5}$.

7. Let ABC be a triangle with $BC = 5$, $CA = 3$, and $AB = 4$. Variable points P, Q are on segments AB, AC, respectively such that the area of APQ is half of the area of ABC. Let x and y be the lengths of perpendiculars drawn from the midpoint of PQ to sides AB and AC, respectively. Find the range of values of $2y + 3x$.
Answer: $[6, 6.5]$
Solution: Use coordinates where you place $A = (0, 0)$, $B = (0, 4)$, $C = (3, 0)$. $P = (a, 0)$, $Q = (0, b)$ where $a \in [0, 3]$ and $b \in [0, 4]$. Then the area condition gives us that $ab = 6$. Note $M = \frac{1}{2}(a, b) = (x, y)$. Thus $x \in [0, 3/2]$, $y \in [0, 2]$ and $xy = 3/2$. Using $y = \frac{3}{2x}$, we have $x \geq \frac{3}{4}$. Thus we need to find the range of

$$\frac{3}{x} + 3x \qquad x \in [3/4, 3/2]$$

The minimum, by AM-GM is 6 for $x = 1$ which is in the interval. Checking the endpoints, we get values of 6.25 and 6.5. Thus the range is $[6, 6.5]$.

8. ABC is an isosceles right triangle with right angle B and $AB = 1$. ABC has an incenter at E. The excircle to ABC at side AC is drawn and has center P. Let this excircle be tangent to AB at R. Draw T on the excircle so that RT is the diameter. Extend line BC and draw point D on BC so that DT is perpendicular to RT. Extend AC and let it intersect with DT at G. Let F be the incenter of CDG. Find the area of $\triangle EFP$.
Answer: $1 + \sqrt{2}$
Solution: I was going to make it harder but I gave up, because I can't do geometry. There are a ton of similar triangles in this figure (i.e. (ABC, CDG), (CQF, PQD), (PCQ, DFQ), (DFC, FQC), (BEP, DFP), (BEP, CEP), (PDQ, CDF)). We can probably make harder problems using similar triangles in other ways. But, back to the solution: The radius of an excircle at side c is given by $\dfrac{2K}{a+b-c}$, where K is the area of the triangle, so we have $PR = \dfrac{1}{2-\sqrt{2}} = 1 + \dfrac{\sqrt{2}}{2}$. Additionally, we have $PR = PT = TD$, so $PD = \sqrt{2}PR$, and $CD = 2PR - 1$, and solving for both gives us $PD = CD = 1 + \sqrt{2}$. Then, we notice similar triangles PDC, APC, so $AP = CP, PD = CD$, we have $PD/PC = PC/AC$, so $PC^2 = AC \cdot PD = 2 + \sqrt{2}$. We have $\angle APE = \pi/8$, so then, we have $AE = PA\tan(\pi/8) = PC\tan(\pi/8)$. Additionally, by similar triangles, we have $AE = EC, CF = (1+\sqrt{2})(EC)$, so then, we can solve for the area of EPF as $(EC + CF)(PC)/2 = EC(2+\sqrt{2})(PC)/2 = PC^2\tan(\pi/8)(2+\sqrt{2})/2 = (2+\sqrt{2})(\tan(\pi/8))(2+\sqrt{2})/2$. Doing some simplifying will give us $1+\sqrt{2}$ as our answer.

9. Let ABC be a triangle. Points D, E, F are on segments BC, CA, AB, respectively. Suppoe that $AF = 10$, $FB = 10$, $BD = 12$, $DC = 17$, $CE = 11$, and $EA = 10$. Suppose that the circumcircles of $\triangle BFD$ and $\triangle CED$ intersect again at X. Find the circumradius of $\triangle EXF$.
Answer: $5\sqrt{2}$
Solution: One can show that $AEXF$ is cyclic using angle chasing. So it suffices to compute the circumradius of $\triangle AEF$, R'. Also $\angle BAC = 90°$. Noting that EAF is a right triangle, $R' = \dfrac{1}{2}EF = \boxed{5\sqrt{2}}$.

10. Let D, E, and F be the points at which the incircle, ω, of $\triangle ABC$ is tangent to BC, CA, and AB, respectively. AD intersects ω again at T. Extend rays TE, TF to hit line BC at E', F', respectively. If $BC = 21$, $CA = 16$, and $AB = 15$, then find $\left|\dfrac{1}{DE'} - \dfrac{1}{DF'}\right|$.
Answer: $\dfrac{1}{110}$

Solution: Let the tangent to the incircle at T intersect BC at X. An important lemma states that because AE, AF are tangents and A, T, D are collinear that T, E, D, F is a harmonic quadrilateral. Using point, T, we project this quadrilateral onto line BC to get that X, F', D, E' form a harmonic 4-tuple. Thus $\dfrac{F'D}{DE'} = \dfrac{XF'}{XE'} = \dfrac{XD - F'D}{XD + DE'}$. It follows that the desired quantity is equal to $\dfrac{2}{DX}$. Another important lemma states that X, B, D, C form a harmonic 4-tuple. Thus $\dfrac{XB}{XC} = \dfrac{BD}{DC} = \dfrac{XD - BD}{XD + DC}$. It follows that $\dfrac{2}{DX} = \left|\dfrac{1}{BD} - \dfrac{1}{DC}\right|$. It follows that the answer is $\dfrac{1}{10} - \dfrac{1}{11} = \boxed{\dfrac{1}{110}}$.

P1. Suppose a convex polygon has a perimeter of 1. Prove that it can be covered with a circle of radius $1/4$.

Solution: Let A be one of the vertices of the polygon. If we walk around the perimeter of the polygon, the total length of our walk will be one. Let B be the halfway point of the walk, and let the midpoint of AB be O. We claim if the center of the circle is at O, then it will cover the whole polygon. Assume, for contradiction, that there is a point P on the boundary which is not covered by the disk. Then, $OP > 1/4$. Now, consider the walk of length $1/2$ from A to B which passes through P. Then the walk from A to P is at least the length of AP, and similarly for P to B and PB. Since the walk from A to P to B is $1/2$, we have $AP + PB \leq 1/2$. Then, consider the reflection of P about point O and denote it P'. Then, $AP' = BP$ because of the reflection, and $PP' = 2OP > 1/2$. But, we then do not have the triangle inequality satisfied in triangle PAP', since $AP + AP' = AP + BP \leq 1/2$, but $PP' > 1/2$ which is the contradiction that we wanted.

P2. From a point A construct tangents to a circle centered at point O, intersecting the circle at P and Q respectively. Let M be the midpoint of PQ. If K and L are points on circle O such that K, L, and A are collinear, prove $\angle MKO = \angle MLO$.

Solution: Since $AP = AQ$, the median AM is also the perpendicular bisector of PQ. But the perpendicular bisector of any chord passes through the center of the circle, so O, M, A are collinear. Note that

$$\triangle OPA \sim \triangle PMA \quad \Longrightarrow \quad \frac{AO}{AP} = \frac{AP}{AM} \quad \Longrightarrow \quad AM \cdot AO = AP^2.$$

Also, by Power of a Point, $AP^2 = AL \cdot AK$. So $AM \cdot AO = AL \cdot AK$, and by the converse of Power of a Point, $KMOL$ is cyclic. It follows that $\angle MKO = \angle MLO$.