1. Find the value of a satisfying

$$a+b=3$$
$$b+c=11$$
$$c+a=61$$

Answer: $\frac{53}{2}$

Solution: Adding the three equations yields $2a+2b+2c=75$, equivalent to $a+b+c=\frac{75}{2}$. Because $b+c=11$, we conclude $a=(a+b+c)-(b+c)=\frac{75}{2}-11=\boxed{\frac{53}{2}}$.

2. A point P is given on the curve $x^4+y^4=1$. Find the maximum distance from the point P to the origin.
Answer: $\sqrt[4]{2}$
Solution: The distance from the origin to any point (x,y) on the curve is $\sqrt{x^2+y^2}$. Because x^2 and y^2 are positive real numbers, we apply AM-QM to conclude $\frac{x^2+y^2}{2}\le\sqrt{\frac{x^4+y^4}{2}}=\sqrt{\frac{1}{2}}$. We thus get, by multiplying by two and taking the square root,

$$\sqrt{x^2+y^2}\le\sqrt{2\cdot\sqrt{\frac{1}{2}}}=\boxed{\sqrt[4]{2}}.$$

Indeed, $x=y=\sqrt[4]{\frac{1}{2}}$ yields the desired equivalence.

3. Evaluate

$$\lim_{x\to 0}\frac{\sin 2x}{e^{3x}-e^{-3x}}$$

Answer: $\frac{1}{3}$
Solution: Recall the definition of the derivative at 0 as $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}$. Then,

$$\lim_{x\to 0}\frac{\sin 2x}{e^{3x}-e^{-3x}}=\lim_{x\to 0}\frac{\sin 2x}{x}\cdot\left(\lim_{x\to 0}\frac{e^{3x}}{x}-\frac{e^{-3x}}{x}\right)^{-1}=2\cdot(3+3)^{-1}=\boxed{\frac{1}{3}}.$$

4. Given a complex number z satisfies $\text{Im}(z) = z^2 - z$, find all possible values of $|z|$.

Answer: $\{0, \frac{\sqrt{2}}{2}, 1\}$

Solution: Setting $z = \text{Re}(z) + \text{Im}(z)i$, where $\text{Re}(z)$ and $\text{Im}(z)$ are real numbers, we find $\text{Im}(z) = \text{Re}(z)^2 - \text{Im}(z)^2 - \text{Re}(z) + (2\text{Re}(z)\text{Im}(z) - \text{Im}(z))i$. Inspecting the imaginary part, we notice either $2\text{Re}(z) - 1 = 0$ or $\text{Im}(z) = 0$. Because we have the further condition imposed by the real part $\text{Im}(z)^2 - \text{Im}(z) = \text{Re}(z)^2 - \text{Re}(z)$, we find all solutions $\frac{1}{2} - \frac{1}{2}i$, 0, and 1. Thus, $|z| \in \boxed{\{0, \frac{\sqrt{2}}{2}, 1\}}$.

5. Suppose that $c_n = (-1)^n(n+1)$. While the sum $\sum_{n=0}^{\infty} c_n$ is divergent, we can still attempt to assign a value to the sum using other methods. The Abel Summation of a sequence, a_n, is $\text{Abel}(a_n) = \lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n$. Find $\text{Abel}(c_n)$.

Answer: $\frac{1}{4}$

Solution: $\sum_{n=0}^{\infty} c_n x^n = \frac{1}{(1+x)^2}$ for $|x| < 1$. Taking x to 1 gives $\boxed{\frac{1}{4}}$.

6. The *minimal polynomial* of a complex number r is the unique polynomial with rational coefficients of minimal degree with leading coefficient 1 that has r as a root. If f is the minimal polynomial of $\cos\frac{\pi}{7}$, what is $f(-1)$?

Answer: $-\frac{7}{8}$

Solution: Recall $x = \cos\frac{\pi}{7} = \frac{1}{2}\left(e^{i\pi/7} + e^{-i\pi/7}\right)$. Additionally, $8x^3 = e^{3i\pi/7} + 3e^{i\pi/7} + 3e^{-i\pi/7} + e^{-3i\pi/7}$, $4x^2 = e^{2i\pi/7} + 2 + e^{-2i\pi/7}$, and $2x = e^{i\pi/7} + e^{-i\pi/7}$. Thus,

$$g(x) = 8x^3 - 4x^2 - 4x + 1 = e^{3i\pi/7} - e^{2i\pi/7} + e^{i\pi/7} - 1 + e^{-i\pi/7} - e^{-2i\pi/7} + e^{-3i\pi/7}.$$

We claim the right hand side in the above equation has numeric value 0, and this follows from $e^{i\pi/7}$ being a fourteenth root of unity but not a seventh root of unity (or -1).

$$x^14 - 1 = (x^7 - 1)(x^7 + 1) = (x^7 - 1)(x + 1)(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)$$

whence we determine $e^{3i\pi/7} \cdot (e^{3i\pi/7} - e^{2i\pi/7} + e^{i\pi/7} - 1 + e^{-i\pi/7} - e^{-2i\pi/7} + e^{-3i\pi/7}) = 0$. Then, applying the rational root test, we determine $g(x)$ has no factors of degree 1 and therefore is irreducible over the rationals. Thus, the minimal polynomial of $\cos\frac{\pi}{7}$ is $f(x) = x^3 - \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{8}$. Plugging in $x = -1$ yields our answer of $f(-1) = \boxed{-\frac{7}{8}}$.
Note that it is no coincidence the numerator's only factor is the odd prime present in the denominator of $\cos\frac{\pi}{k}$.

7. If x, y are positive real numbers satisfying $x^3 - xy + 1 = y^3$, find the minimum possible value of y.

Answer: $\frac{1}{3}\sqrt[3]{29 - 4\sqrt{7}}$

Solution: Consider the intersection of cubic function $f(x) = x^3 + 1$ and linear function $g(x) = y \cdot x + y^3$. In order to minimize the value of y while these functions still have an intersection in the First Quadrant, the linear function must be tangent to $f(x)$. Thus, for some value of intersection $x = a$, we have $f(a) = g(a)$ and $f'(a) = g'(a)$. Therefore, $a^3 + 1 = ya + y^3$ and $3a^2 = y$. Plugging the latter into the former,

$$a^3 + 1 = 3a^3 + 27a^6$$

$$27a^6 + 2a^3 - 1 = 0$$

$$a^3 = \frac{2\sqrt{7} - 1}{27}$$

$$a = \frac{1}{3}\sqrt[3]{2\sqrt{7} - 1} \text{ and } y = \boxed{\frac{1}{3}\sqrt[3]{29 - 4\sqrt{7}}}.$$

8. Billy is standing at $(1, 0)$ in the coordinate plane as he watches his Aunt Sydney go for her morning jog starting at the origin. If Aunt Sydney runs into the First Quadrant at a constant speed of 1 meter per second along the graph of $x = \frac{2}{5}y^2$, find the rate, in radians per second, at which Billy's head is turning clockwise when Aunt Sydney passes through $x = 1$.

Answer: $\frac{4}{\sqrt{65}}$

Solution:

Let t be the amount of time that has passed, (x, y) be the coordinates of Aunt Sydney's positions and θ be the angle Billy's head makes with the negative x-axis. ($\theta = 0$ initially.) We wish to find $\frac{d\theta}{dt}$ when $x = 1$. By considering the triangle formed by Billy, Aunt Sydney, and the point $(x, 0)$ we get that $\theta = \tan^{-1}\left(\frac{y}{1 - \frac{2}{5}y^2}\right)$. $\frac{d\theta}{dy} = \frac{1 + \frac{2}{5}y^2}{(1 - \frac{2}{5}y^2)^2 + y^2}$. At $x = 1$, $\frac{d\theta}{dx} = \frac{4}{5}$.

Let the distance Aunt Sydney runs be $s = \int_0^y \sqrt{1 + \frac{16t^2}{25}}dt$. Then $1 = \frac{ds}{dt} = \frac{dx}{dt}\sqrt{1 + \frac{16y^2}{25}}$, so $\frac{dx}{dt} = \frac{1}{\sqrt{1 + \frac{16x^2}{25}}}$. At $x = 1$, $\frac{dx}{dt} = \frac{\sqrt{5}}{\sqrt{13}}$. Now $\frac{d\theta}{dt} = \frac{d\theta}{dx}\frac{dx}{dt} = \frac{4}{5}\frac{\sqrt{5}}{\sqrt{13}} = \boxed{\frac{4}{\sqrt{65}}}$.

9. Evaluate the integral

$$\int_0^1 \sqrt{(x-1)^3 + 1} + x^{2/3} - (1-x)^{3/2} - \sqrt[3]{1 - x^2}\,dx$$

Answer: $\frac{1}{5}$

Solution: We first show that $\int_0^1 \sqrt{(x-1)^3 + 1} - \sqrt[3]{1 - x^2}\,dx = 0$. This is easily obtained by considering the graph of $y^2 = (x-1)^3 + 1$ from $(0, 0)$ to $(1, 1)$. From this, we have $\int_0^1 \sqrt{(x-1)^3+1}\,dx + \int_0^1 \sqrt{1-y^2}+1\,dy = 1$. Combining integrals yields the desired equivalence. Then, evaluating the two remaining terms, we obtain an answer of

$$\int_0^1 x^{2/3} - (1-x)^{3/2} = \frac{3}{5} - \frac{2}{5} = \boxed{\frac{1}{5}}.$$

10. Let the class of functions f_n be defined such that $f_1(x) = |x^3 - x^2|$ and $f_{k+1}(x) = |f_k(x) - x^3|$ for all $k \geq 1$. Denote by S_n the sum of all y-values of $f_n(x)$'s "sharp" points in the First Quadrant. (A "sharp" point is a point for which the derivative is not defined.) Find the ratio of odd to even terms,

$$\lim_{k\to\infty} \frac{S_{2k+1}}{S_{2k}}$$

Answer: $\frac{1}{7}$

Solution: Note that there exist sharp points of $f_n(x)$ at all $x = 1/1, 1/2, 1/3, \cdots, 1/n$ and that (a different) half of them are 0 for odd or even values of n. We then wish to find A/B where $A = \sum_{k=1}^{\infty} \frac{1}{(2k)^3}$ and $B = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3}$. Noting that $8A = \sum_{k=1}^{\infty} \frac{1}{k^3} = A + B$, we obtain $7A = B$ or $A/B = \boxed{1/7}$.

P1. Prove that for all positive integers m and n,

$$\frac{1}{m}\cdot\binom{2n}{0} - \frac{1}{m+1}\cdot\binom{2n}{1} + \frac{1}{m+2}\cdot\binom{2n}{2} - \cdots + \frac{1}{m+2n}\cdot\binom{2n}{2n} > 0$$

Solution: Consider the integral

$$\int_0^1 x^{m-1}(x-1)^{2n}\,dx$$

The integrand $x^{m-1}(x-1)^{2n}$ is equal to $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}x^{m+k-1}$. Thus, the antiderivative of the integrand is equal to $\sum_{k=0}^{2n}\frac{(-1)^k}{m+k}\binom{2n}{k}x^{m+k}$, and the given integral evaluates to the expression in the question. Then, because x^{m-1} and $(x-1)^{2n}$ are always positive functions, the area beneath the curve must be positive as well. Thus, the sum is greater than 0, as desired.

P2. If $f(x) = x^n - 7x^{n-1} + 17x^{n-2} + a_{n-3}x^{n-3} + \cdots + a_0$ is a real-valued function of degree $n > 2$ with all real roots, prove that no root has value greater than 4 and at least one root has value less than 0 or greater than 2.

Solution: If the roots are $r_1, \ldots, r_n$, then $r_1 + \cdots + r_n = 7$ and $(r_1)^2 + \cdots + (r_n)^2 = 15$. Clearly, no root may have value greater than 4, as $(r_1)^2 + \cdots + (r_n)^2 - (r_k)^2 < -1$ is an impossibility. Furthermore, we may conclude

$$\sum_{k=1}^{n}(r_k-1)^2 = \sum_{k=1}^{n}(r_k)^2 - 2\sum_{k=1}^{n}r_k + \sum_{k=1}^{n}1 = 15 - 14 + n = n+1.$$

If all roots are between 0 and 2, then all values of $(r_k - 1)^2$ are below 1, so $\sum (r_k - 1)^2 < n$, a contradiction. Thus, both results have been realized.