INDIVIDUAL ROUND SOLUTIONS

1. Let d be the common difference between each term of the progression. Then we have

$$5(a_1 - 12d) = 6(a_1 - 18d)$$
$$5a_1 - 60d = 6a_1 - 108d$$
$$48d = a_1$$

Thus, $a_{49} = a_1 - 48d = 0$ and $a_{50} = -d$ and thus $n = \boxed{50}$.

2. For any number n, let $p_1^{e_1} p_2^{e_2} \dots$ be the prime factorization. The number of divisors of n is given by $(e_1+1)(e_2+1)\dots$. We have that 28 factors into $2 \cdot 2 \cdot 7$, thus, in order to minimize our number, we want $2^6 3^1 5^1$ (the lowest three prime numbers, with higher exponents on lower numbers). This is equal to $\boxed{960}$

3. First, observe that

$$\frac{k}{n(n+1)(n+2)\cdots(n+k)} = \frac{k}{k}\left(\frac{1}{n(n+1)(n+2)\cdots(n+k-1)} - \frac{1}{(n+1)(n+2)\cdots(n+k)}\right)$$

Thus,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{k}{n(n+1)(n+2)\cdots(n+k)} \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)(n+2)\cdots(n+k-1)} - \frac{1}{(n+1)(n+2)\cdots(n+k)}\right) \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)(n+2)\cdots(n+k-1)} - \frac{1}{(n+1)(n+2)\cdots(n+k)}\right) \\
&= \left[\left(\frac{1}{(1)(2)\dots(k)} - \frac{1}{(2)(3)\cdots(k+1)}\right) + \left(\frac{1}{(2)(3)\dots(k+1)} - \frac{1}{(3)(4)\cdots(k+2)}\right) + \dots\right] \\
&= \frac{1}{(1)(2)\cdots(k)} \\
&= \frac{1}{k!}
\end{aligned}$$

When $k = 2012$, we have that the largest prime factor of the reciprocal is 2011.

4. In total, Tyler has $\frac{4025 \cdot 4026}{2}$ ways to win. Out of these ways, we have that $\frac{2012 \cdot 2013}{2}$ of these ways are when the first dice is less than $\frac{4025}{2}$.Thus, given that he won, the probability that the number on the first dice was less than $\frac{4025}{2}$ is

$$\frac{\frac{2012 \cdot 2013}{2}}{\frac{4025 \cdot 4026}{2}} = \frac{1006}{4025}.$$

5. To find the remainder of our number when divided by 60, we must find the remainder mod 3, 4, and 5. Obviously, $3 \uparrow\uparrow (3 \uparrow\uparrow (3 \uparrow\uparrow 3)) = 0 \mod 3$. To find $3 \uparrow\uparrow (3 \uparrow\uparrow (3 \uparrow\uparrow 3)) \mod 4$, we note that the exponent of 3 is certainly odd, and thus $3 \uparrow\uparrow (3 \uparrow\uparrow (3 \uparrow\uparrow 3)) = 3 \mod 4$.
Meanwhile,

$$3 \uparrow\uparrow (3 \uparrow\uparrow (3 \uparrow\uparrow 3)) = 3^{3 \uparrow\uparrow ((3 \uparrow\uparrow (3 \uparrow\uparrow 3)) - 1)}.$$

By similar reasoning, this exponent is equal to 3 mod 4, and thus $3 \uparrow\uparrow (3 \uparrow\uparrow (3 \uparrow\uparrow 3)) = 2 \mod 5$. Using the Chinese Remainder Theorem, we then obtain that $3 \uparrow\uparrow (3 \uparrow\uparrow (3 \uparrow\uparrow 3)) = 27 \mod 60$.

6. We draw all the perpendiculars from the incenters to touch the edges of their respective triangles. Then using congruent triangles, we have $DR - RB = DA - AB = 10$, and $DS - SB = DC - CB = 2$. Thus $RS = 4$.

7. Note that since $(c+d)$ and $(b+c)$ are prime, $c = 2$. This is the smallest prime number. Obviously, the largest prime number given is $(a+b+c+18+d)$. So, we have the difference equal to $(a+b+18+d)$. Since d is prime, $2010+18+d \implies d \equiv 1 \pmod 3$ or $d \equiv 2 \pmod 3$. Since $(d+2)$ is prime, d must be equivalent to 2 (mod 3). Now, given that $d \leq 50$, we can list out the prime numbers that are equivalent to 2 (mod 3).
These are: $2, 5, 11, 17, 23, 29, 41, 47$
We can eliminate all numbers ending with 3 and 7 because $(a+b+c+18+d)$ or $(a+b+c+18-d)$ will be divisible by 5 if d ends with 3 or 7. So, we are left with $5, 11, 29$, and 41. With four numbers left, we can easily check if $2028+d$ and $2028-d$ are prime. If $d = 5$, then $2028 - 5 = 2023$ is not prime because

$202 - 2(3) = 196$. $19 - 2(6) = 7$, which is divisible by 7. Thus, 2023 is divisible by 7. If $d = 11$, then we have $2028 - 11 = 2017$, which is divisible by 3. If $d = 29$, then we have $2028 - 29 = 1999$, which is prime. But $2028 + 29 = 2057$ is not because $2 + 5 = 0 + 7 \implies 2057$ is divisible by 11. Thus, $d = 41$, and our largest prime number is 2069.

Thus, the answer is $2069 - 2 = \boxed{2067}$.

8. Let $f(n, m)$ be the expected number of flips we need in order to get n consecutive heads given that we have already flipped m heads. We have the base cases $f(n, n) = 0$, and the recurrence

$$f(n, m) = \frac{1}{2}f(n, m+1) + \frac{1}{2}f(n, 0) + 1$$

We get this recurrence because with probability $\frac{1}{2}$, we get one more head, with probability $\frac{1}{2}$ we have to start over with zero heads. We want to solve this recurrence in general for any n, m. Notice that n stays fixed throughout the recurrence, and $f(n, 0)$ appears in each term of the recurrence.
We have $f(n, n) = 0, f(n, n-1) = \frac{1}{2}f(n, 0)+1, f(n, n-2) = \frac{1}{2}\left(\frac{1}{2}f(n, 0) + 1 + f(n, 0)\right)+1 = \frac{3}{4}f(n, 0)+\frac{3}{2}$, and in general, $f(n, n-k) = \frac{2^k-1}{2^k}f(n, 0) + \frac{2^k-1}{2^{k-1}}$ (which can be proved by induction). So, solving for $f(n, 0)$, we have $f(n, 0) = \frac{2^n-1}{2^n}f(n, 0) + \frac{2^n+1}{2^{n-1}}$, which gives us $f(n, 0) = 2^{n+1} - 2$. Plugging this in for general gives us

$$f(n, n-k) = \frac{2^k - 1}{2^k}(2^{n+1} - 2) + \frac{2^k - 1}{2^k}$$

$$= \frac{(2^k - 1)(2^{n+1} - 2) + 2^{k+1} - 2}{2^k} = \frac{2^{n+k+1} - 2^{n+1} - 2^{k+1} + 2 + 2^{k+1} - 2}{2^k} = 2^{n+1} - 2^{n-k+1}$$

So, we have in general, $f(n, m) = 2^{n+1} - 2^{m+1}$, so plugging in our values, we get

$$2^{60} - 2^{28} = 2^{28}(2^{16} + 1)(2^8 + 1)(2^4 + 1)(2^2 + 1)(2^2 - 1)$$

and the sum of the prime factors is $2 + 65537 + 257 + 17 + 5 + 3 = 65821$

9. The solution to this problem just involves a case-by case analysis.

Let's label the left-pin pin-B and the right pins $A_1, A_2, A_3 \dots$ up to A_k

Now when $k = 1$, there is no winning move (If Jing Jing takes A_1, Soumya takes B, if Soumya takes A_1, Jing Jing takes B)

Then for $k =$ 2,3, or 4, the winning move is of course to take all pins except for A_1 so clearly Jing Jing leaves Soumya in a losing position as above.

If $k = 5$, there are a few possible moves. If Jing Jing takes any pins off the end (eg $A_1 \dots A_i$ or $A_j \dots A_k$), then she reduces the situation to be equivalent to $k =$ 2,3, or 4 and so Soumya can win as above.

If Jing Jing takes A_2 and A_3, Soumya can take A_4 and A_5 leaving the position a losing position. If Jing Jing takes just A_3, then Soumya takes just pin B leaving two identical piles. Whenever there are two identical piles, the position is losing, since Soumya can then mimic Jing Jing's moves. Finally if Jing Jing takes A_2, A_3, and A_4, Soumya can take any one of the remaining piles of 1 again leaving two identical 1-piles.

This shows that $k = 5$ is a losing position.

Then $k = 6, 7, 8$ are all winning positions since they can be reduced to $k = 5$ by removing pins from the end of the A's

For $k = 9$, we can go through all the possible moves.

Again, removing pins off the end leaves a winning position as we've already identified.

Taking just A_2, leaves two 1-pin piles and a 7-pin pile. Soumya can respond by taking out the middle of the center-pin pile to leave two sets of identical piles. This is a losing-position since Soumya can now play with a mirroring strategy (or rather a tweedle-dee tweedle-dum strategy as it's called in "Winning Ways for your Mathematical Plays" by Elwyn Berlekamp, John Conway, and Richard Guy)

Taking just A_3 leaves a 1-pile a 2-pile and a 6-pile. By removing A_6,A_7, A_8, we again leave two two-pin piles and two 1-pin piles

For the remaining possible moves here's a list of counter-moves which leave a symmetric tweedle-dee tweedle-dum losing position

Jing Jing's Move	Soumya's response
A_4	A_8
A_5	B
A_2, A_3	A_6,A_7
A_3, A_4	A_7,A_8
A_2,A_3,A_4	A_7
A_3,A_4,A_5	A_8
A_4,A_5,A_6	B

In fact I left out the A_4, A_5 case. You can verify for yourself that A_6, A_7 is a winning response move (leaving piles of size 1, 2, and 3)

We have successfully seen that $k = 9$ is a losing position so then $k = 10$ must be a winning position since we can pull one off the end of the 10-string and leave our opponent in a losing position.

Then the winning positions for $k < 10$ are $k \in \{2, 3, 4, 6, 7, 8, 10\}$

By the way, if you think $k \neq 1 \pmod 4$ is the solution for all $k \in \mathbb{N}$, consider the $k = 13$ case. What if Jing Jing starts by removing A_4,A_5 leaving groups of sizes 1,3, and 8. Does Soumya have a winning response?

10. To make this problem easier to understand, we first draw a triangular grid. Drawing this grid allows us to just draw the beam as a straight line connecting two different vertices. We also keep track of where the original vertex is by reflecting it along when drawing the grid.

Following the pattern downwards, take the equilateral triangle of length l. We can see that it will take $2l - 3$ reflections to reach a vertex on the bottom level. In addition, we can see that the a copy of the original vertex starts at the $(2l) \pmod 3$ from the bottomleft most vertex, and repeats every three triangles. Thus, for this specific triangle, the side length of the triangle is 70, and the original vertex appears at triangles $2, 5, 8, \ldots 65, 68$.

By symmetry, we only consider the left half the the triangle. In order for the beam to travel back to the original vertex, it must not intersect any other vertex on its way (otherwise it would escape prematurely). Thus, suppose i is the position of the vertex. We can reach this vertex only when $\gcd(70, i) = 1$, since if it was greater than one, that would imply it intersected another vertex somewhere higher up. Maximizing the distance requires us to try to take our vertex as close to the left side as possible. Testing values, we see that $2, 5, 8$ don't work, so we have the vertex at 11. Minimizing the distance requires us to take the vertex as close to the middle as possible. Testing values, we see $35, 32$ don't work, so we have our vertex at 29.

Calculating distances can now be done with law of cosines. We have $M^2 = 70^2 + 11^2 - 70 \cdot 11, m^2 = 70^2 + 29^2 - 70 \cdot 29$. Subtracting, we get $M^2 - m^2 = 11^2 - 29^2 - 70(11 - 29) = (11 + 29 - 70)(11 - 29) = -30 \cdot -18 = \boxed{540}$.