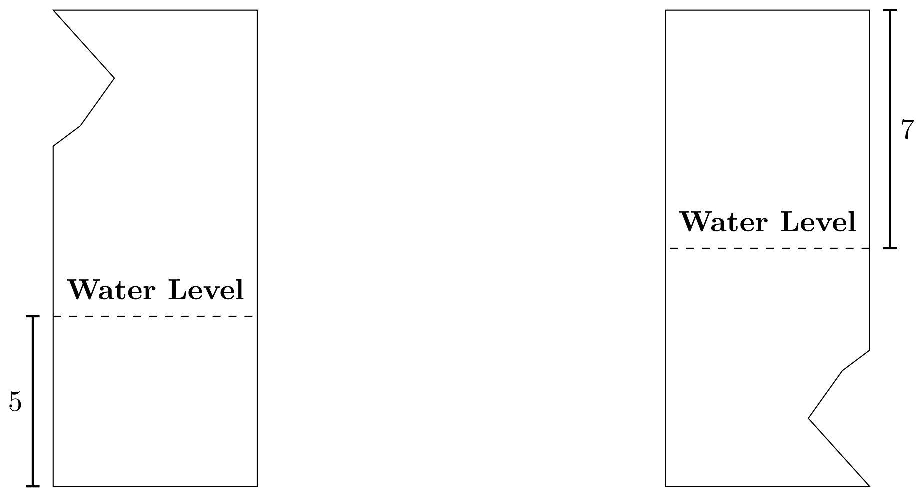
1. Suppose $a_1 \cdot 2 = a_2 \cdot 3 = a_3$ and $a_1 + a_2 + a_3 = 66$. What is a_3?

   Answer: 36

   Solution: We see that $a_1 = \frac{a_3}{2}$, $a_2 = \frac{a_3}{3}$, so $a_1 + a_2 + a_3 = \frac{a_3}{2} + \frac{a_3}{3} + a_3 = \frac{11a_3}{6} = 66$, so $a_3 = \boxed{36}$.

2. Ankit buys a see-through plastic cylindrical water bottle. However, in coming home, he accidentally hits the bottle against a wall and dents the top portion of the bottle (above the 7 cm mark). Ankit now wants to determine the volume of the bottle. The area of the base of the bottle is 20 cm^2. He fills the bottle with water up to the 5 cm mark. After flipping the bottle upside down, he notices that the height of the empty space is at the 7 cm mark. Find the total volume (in cm^3) of this bottle.



   Answer: 240

   Solution: The volume of the empty space in the second bottle diagram is equal to the volume of the empty space in the first diagram, and similarly with the filled space. The volume of the empty space in the second diagram is $7 \times 20 = 140$, so the volume of the empty space in the first bottle is 140 cm^3. The volume of the filled space in the bottle in the first diagram is $5 \times 20 = 100cm^3$. Thus, the total area of the bottle is $140 + 100 = \boxed{240}$.

3. If P is a quadratic polynomial with leading coefficient 1 such that $P(1) = 1$, $P(2) = 2$, what is $P(10)$?

   Answer: 82

   Solution 1: Let $P(x) = x^2 + bx + c$. We have $b + c = 0$, and $2b + c = -2$. Solving for b and c, we have $b = -2$ and $c = 2$. Thus, $P(x) = x^2 - 2x + 2$, so $P(10) = 10^2 - 2(10) + 2 = 100 - 20 + 2 = 82$.

   Solution 2: Alternatively, one can note that $P(x) = x + (x-1)(x-2)$ so $P(10) = 10 + 9 \cdot 8 = 82$.

4. Let ABC be a triangle with $AB = 1$, $AC = 3$, and $BC = 3$. Let D be a point on BC such that $BD = \frac{1}{3}$. What is the ratio of the area of BAD to the area of CAD?

   Answer: $\frac{1}{8}$

Solution: Let h be the height of the triangle with respect to BC. We have that $DC = 9 - \frac{1}{3} = \frac{8}{3}$, so letting $[\cdot]$ denoting area, we have

$$\frac{[BAD]}{[CAD]} = \frac{\frac{1}{2} \cdot BD \cdot h}{\frac{1}{2} \cdot DC \cdot h} = \frac{\frac{1}{2} \cdot \frac{1}{3}h}{\frac{1}{2} \cdot \frac{8}{3}h} = \boxed{\frac{1}{8}}.$$

as desired.

5. A coin is flipped 12 times. What is the probability that the total number of heads equals the total number of tails? Express your answer as a common fraction in lowest terms.

   Answer: $\mathbf{\frac{231}{1024}}$

   Solution: If the number of heads equals the number of tails, then the number of heads is 6 and the number of tails is 6. Therefore, we only need to figure out how to arrange the 12 coins in different ways. The number of ways to do this is $\binom{12}{6} = 924$. Dividing by 2^{12}, the total number of possible outcomes, we obtain, $\boxed{\frac{231}{1024}}$.

6. Moor pours 3 ounces of ginger ale and 1 ounce of lime juice in cup A, 3 ounces of lime juice and 1 ounce of ginger ale in cup B, and mixes each cup well. Then he pours 1 ounce of cup A into cup B, mixes it well, and pours 1 ounce of cup B into cup A. What proportion of cup A is now ginger ale? Express your answer as a common fraction in lowest terms.

   Answer: $\mathbf{\frac{13}{20}}$

   Solution: The 1 ounce that Moor pours from cup A into cup B has a ginger ale to lime juice ratio of 3 to 1, so he pours $\frac{3}{4}$ ounces of ginger ale into cup B and $\frac{1}{4}$ ounces of lime juice into cup B. Therefore, cup A now has $3 - \frac{3}{4} = \frac{9}{4}$ ounces of ginger ale and $1 - \frac{1}{4} = \frac{3}{4}$ ounces of lime juice, and cup B now has $1 + \frac{3}{4} = \frac{7}{4}$ ounces of ginger ale and $3 + \frac{1}{4} = \frac{13}{4}$ ounces of lime juice in cup B. The one ounce that he pours from cup B to cup A has a ginger ale to lime juice ratio to $\frac{7}{4} : \frac{13}{4} = 7 : 13$, so Moor pours $\frac{7}{20}$ ounces of ginger ale into cup A. Hence cup A has

$$\frac{9}{4} + \frac{7}{20} = \frac{52}{20} = \frac{13}{5}$$

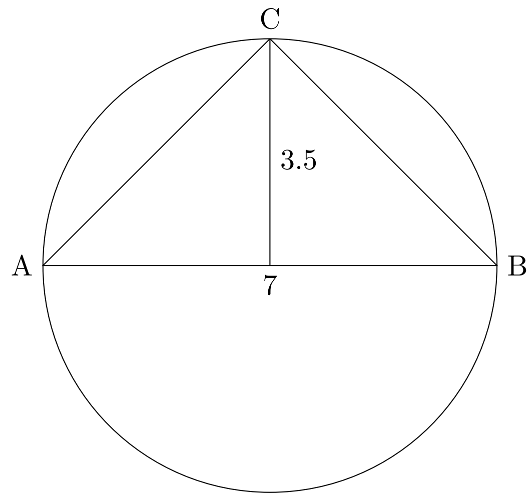
   so the proportion is

$$\frac{13}{5} \times \frac{1}{4} = \boxed{\frac{13}{20}}.$$

7. Determine the maximum possible area of a right triangle with hypotenuse 7. Express your answer as a common fraction in lowest terms.

   Answer: $\mathbf{\frac{49}{4}}$

   Solution: If we inscribe a right triangle in a circle, then the hypotenuse must equal that circle's diameter because thearc intercepted by a right angle is 180 degrees. Therefore, we must consider the set of all possible right triangles with hypotenuse 7 situated on a diameter of a circle, and the third point somewhere else on that circle. The maximum area then occurs at the point with greatest height above the diameter. This means the height is equal to the radius of the circle. If the hypotenuse is 7, then the area of this triangle is $\frac{1}{2} \cdot \frac{7}{2} \cdot 7 = \boxed{\frac{49}{4}}$.



8. Debbie has six Pusheens: 2 pink ones, 2 gray ones, and 2 blue ones, where Pusheens of the same color are indistinguishable. She sells two Pusheens each to Alice, Bob, and Eve. How many ways are there for her to do so?

   Answer: 21

   Solution: We break this up into two cases. If Alice buys two of the same color (say with 2 pink pusheens), there are 3 ways for Debbie to distribute the remaining ones:

   1. 2 gray Pusheens to Bob and 2 blue Pusheens to Eve.
   2. 2 blue Pusheens to Bob and 2 gray Pusheens to Eve.
   3. 1 gray and 1 blue Pusheen to both Bob and Eve.

   There are 3 ways for Alice to buy pusheens of the same color, since there are 3 colors

   If Alice buys two Pusheens of different colors (say, 1 pink pusheen and 1 gray pusheen), there are 4 ways for Debbie to distribute the remaining ones:

   1. 1 pink pusheen and 1 gray pusheen to Bob, and 2 blue pusheens to Eve.
   2. 1 pink pusheen and 1 gray pusheen to Eve, and 2 blue pusheens to Bob.
   3. 1 pink pusheen and 1 blue pusheen to Bob, and 1 blue 1 gray pusheen to Eve.
   4. 1 pink pusheen and 1 blue pusheen to Eve, and 1 blue 1 gray pusheen to Bob.

   There are $\binom{3}{2}$ ways for Alice to buy two pusheens of different colors. Thus there are $3 \cdot 3 + 3 \cdot 4 = \boxed{21}$ ways for Debbie to sell Pusheens to Alice, Bob, and Eve.

9. How many nonnegative integer pairs (a, b) are there that satisfy $ab = 90 - a - b$?

   Answer: 4

   Solution: Rewriting the equation presented in the problem, we have

   $$ab + a + b + 1 = 91 \implies (a + 1)(b + 1) = 91$$

   Since $91 = 7 \times 13$, and a and b are nonnegative integers, one of the following must hold:

   1. $a + 1 = 13$ and $b + 1 = 7 \implies (a, b) = (12, 6)$

2. $b+1=13$ and $a+1=7 \implies (a,b)=(6,12)$
3. $a+1=91$ and $b+1=1 \implies (a,b)=(90,0)$
4. $b+1=91$ and $a+1=1 \implies (a,b)=(0,90)$

This gives $\boxed{4}$ ordered pairs.

10. What is the smallest positive integer $a_1 \dots a_n$ (where $a_1, \dots, a_n$ are its digits) such that $9 \cdot a_1 \dots a_n = a_n \dots a_1$, where $a_1,\ a_n \neq 0$?

Answer: 1089

Solution: We note that this number must start with 1 and end in 9. If $n=2$, then $19 \times 9 > 100$, a contradiction. If $n=3$, then $9(100+10a_2+9) = 900+10a_2+1 \implies 80a_2+80=0$, a contradiction. Hence $n \geq 4$. If $n=4$, we have $9(1000+100a_2+10a_3+9) = 9000+100a_3+10a_2+1 \implies 890a_2+80=10a_3$. Since $0 \leq a_2 \leq 9$, we must have $a_2=0$, and thus $a_3=8$. Thus, 1089 is the only solution for $n=4$. Since 4 digit numbers are less than 5 digit numbers, $\boxed{1089}$ is the least such solution.

11. Justin is growing three types of Japanese vegetables: wasabi root, daikon and matsutake mushrooms. Wasabi root needs 2 square meters of land and 4 gallons of spring water to grow, matsutake mushrooms need 3 square meters of land and 3 gallons of spring water, and daikon need 1 square meter of land and 1 gallon of spring water to grow. Wasabi sell for \$60 per root, matsutake mushrooms sell for \$60 per mushroom, and daikon sell for \$2 per root. If Justin has 500 gallons of spring water and 400 square meters of land, what is the maximum amount of money, in dollars, he can make?

Answer: \$9,000

Solution: Note that the number of daikon grown will be less than 2; if Justin grew 3 daikon, he could grow 1 matsutake mushroom instead and make \$54 more. Additionally, the price of daikon is so low, that maximizing the total number of wasabi and matsutake grown will maximize the profit. Let x be the number of wasabi root and y be the number of matsutake mushrooms. Then our equations are

$$2x+3y \leq 400$$

$$4x+3y \leq 500.$$

We want to maximize $x+y$, and this is maximized in the equality case for both equations. This occurs when

$$x=50, y=100.$$

12. A *prim* number is a number that is prime if its last digit is removed. A *rime* number is a number that is prime if its first digit is removed. Determine how many numbers between 100 and 999 inclusive are both prim and rime numbers.

Answer: 52

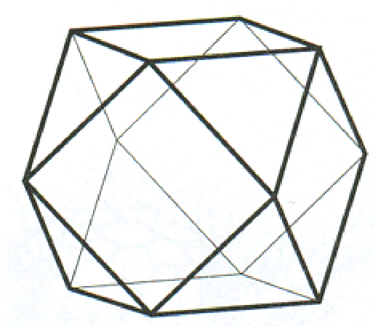
Solution: We need to find three digit numbers whose first two digits are a prime and whose last two digits are a prime. Let us start by examining the possibilities for the middle digit. Since the middle digit is the units digit of a two-digit prime number, it must have a units digit of 1, 3, 7, or 9. Let us write out two tables of primes, one with ending digit 1, 3, 7, and 9, and the other with starting digits 1, 3, 7, and 9.

Last digit 1	Last digit 3	Last digit 7	Last digit 9
11, 31, 41, 61, 71	13, 23, 43, 53, 73, 83	17, 37, 47, 67, 97	19, 29, 59, 79, 89

First digit 1	First digit 3	First digit 7	First digit 9
11, 13, 17, 19	31, 37	71, 73, 79	97

Now we simply pair up the numbers with same last digit with numbers with the same first digit and we get a 3 digit number that is both *prim* and *rime*. Therefore, our final answer is $5 \cdot 4 + 6 \cdot 2 + 5 \cdot 3 + 5 \cdot 1 = \boxed{52}$.

13. Consider a cube. Each corner is the intersection of three edges; slice off each of these corners through the midpoints of the edges, obtaining the shape below. If we start with a $2 \times 2 \times 2$ cube, what is the volume of the resulting solid?



Answer: $\mathbf{\frac{20}{3}}$

Solution: The cube has volume 8. Each sliced-off-corner is a pyramid with base area $\frac{1}{2}$ and height 1, so each corner has volume $\frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6}$. The volume of the solid is therefore $8 - 8 \cdot \frac{1}{6} = \boxed{\frac{20}{3}}$.

14. If a parallelogram with perimeter 14 and area 12 is inscribed in a circle, what is the radius of the circle?

Answer: $\mathbf{\frac{5}{2}}$ **or 2.5**

Solution: If a parallelogram is inscribed in a circle, it is a rectangle. Because opposite angles of a parallelogram are equal, and together the angles intercept arcs totaling $360°$, then each angle intercepts an arc of $180°$. Let x and y be side lengths of adjacent sides on the rectangle. Then, $x + y = 7$ and $xy = 12$, giving us $x = 3$ and $y = 4$, with the diagonal of the rectangle being 5. Because the hypotenuse corresponds to the diameter, the radius is $\boxed{\frac{5}{2}}$.

15. Take a square $ABCD$ of side length 1, and draw $\overline{AC}$. Point E lies on $\overline{BC}$ such that $\overline{AE}$ bisects $\angle BAC$. What is the length of BE?

Answer: $\mathbf{\sqrt{2} - 1}$

Solution 1: Draw F on $\overline{AC}$ so that $\overline{EF}$ is perpendicular to $\overline{AC}$. Then

$$BE = EF = FC,$$

and

$$EC = \sqrt{2}BE.$$

Thus, $1 = (1 + \sqrt{2}) \cdot BE$ and

$$BE = \boxed{\sqrt{2} - 1}.$$

Solution 2: By the angle bisector theorem, we have

$$\frac{AB}{BE} = \frac{AC}{CE} \implies \frac{1}{BE} = \frac{\sqrt{2}}{1 - BE} \implies \sqrt{2}BE = 1 - BE \implies BE = \frac{1}{1 + \sqrt{2}} = \boxed{\sqrt{2} - 1}$$

as desired.

16. How many integer solutions does $f(x) = (x^2 + 1)(x + 2) + (x^2 + 3)(x + 4) = 2017$ have?

Answer: 0

Solution 1: Expanding $f(x)$ gives us $(x^2 + 1)(x + 2) + (x^2 + 3)(x + 4) = x^3 + x + 2x^2 + 2 + x^3 + 3x + 4x^2 + 12 = 2x^3 + 6x^2 + 4x + 14 = 2(x^3 + 3x^2 + 2x + 7)$. Since 2017 is odd and $f(x)$ is even if x is an integer, there is no integer solution for $f(x) = 2017$.

Solution 2: Taking the equation modulo 2, we have

$$(x^2 + 1)x^2 + (x^2 + 1)x^2 \equiv 1 \pmod 2 \implies 0 \equiv 1 \pmod 2$$

a contradiction.

17. Alice, Bob, Carol, and Dave stand in a circle. Simultaneously, each player selects another player at random and points at that person, who must then sit down. What is the probability that Alice is the only person who remains standing?

Answer: $\mathbf{\frac{4}{27}}$

Solution: First, we see that the probability 3 or 4 people remain standing is 0. Then, the probability that Alice and Bob both remain standing is $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{4}{81}$. The probability Alice remains standing is $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$. Then the probability Alice is the only person standing is

$$\frac{8}{27} - 3 \cdot \frac{4}{81} = \boxed{\frac{4}{27}}.$$

18. Let x be a positive integer with a remainder of 2 when divided by 3, 3 when divided by 4, 4 when divided by 5, and 5 when divided by 6. What is the smallest possible such x?

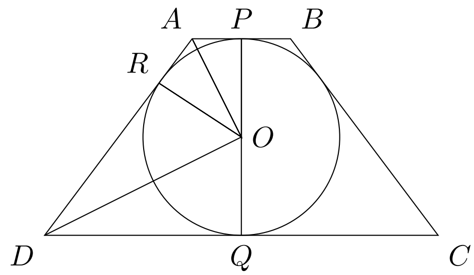
Answer: 59

Solution: $x + 1$ will be divisible by $3, 4, 5$, and 6, and the least common multiple of thse four numbers is 60, so $x = \boxed{59}$.

19. A circle is inscribed in an isosceles trapezoid such that all four sides of the trapezoid are tangent to the circle. If the radius of the circle is 1, and the upper base of the trapezoid is 1, what is the area of the trapezoid?

Answer: 5

Solution 1: Let us draw some extra lines as shown below with labels.



We see that $PAOR$ and $QORD$ are both kites. Then, $\angle POA = \angle AOR$ and $\angle ROD = \angle DOQ$. Therefore, $\angle AOD = \frac{1}{2}\angle POQ = 90$. Therefore, since $\angle ARO = \angle ORD = 90$, triangles $\Delta ARO \sim \Delta ORD$. Since $\Delta PAO \cong \Delta ARO$ and $\Delta ODQ \cong \Delta ORD$, we see that $\Delta PAO \sim \Delta ODQ$. Therefore, since $PA : PO = 1 : 2 = OD : DQ$ and $OD = 1$, we see that $DQ = 2$. Therefore, the trapezoid's longest base has length 4. Finally, the area of the trapezoid is $\frac{4+1}{2} \cdot 2 = \boxed{5}$.

Solution 2: We refer to the diagram in solution 1. Let $CD = y$. Then we have by symmetry, $AP = \frac{AB}{2} = \frac{1}{2} \implies AP = AR = \frac{1}{2}$, $DR = DQ = \frac{y}{2} \implies AD = \frac{y+1}{2}$. Let A' be the orthogonal projection of point A to point CD. Then $A'Q = \frac{1}{2}$, so $DA' = \frac{y-1}{2}$. Using the pythagorean theorem on triangle ADA', we have that $\left(\frac{y-1}{2}\right)^2 + 2^2 = \left(\frac{y+1}{2}\right)^2 \implies y = 4$. Thus, the area of $ABCD$ is $\frac{1}{2} \cdot 2 \cdot (1+4) = \boxed{5}$.

20. Ray is blindfolded and standing 1 step away from an ice cream stand. Every second, he has a $\frac{1}{4}$ probability of walking 1 step towards the ice cream stand, and a $\frac{3}{4}$ probability of walking 1 step away from the ice cream stand. When he is 0 steps away from the ice cream stand, he wins. What is the probability that Ray eventually wins?

Answer: $\mathbf{\frac{1}{3}}$

Solution: Let p_1 be the desired probability. Then we have $p_1 = \frac{1}{4} + \frac{3}{4}p_2$, where p_2 is the probability Ray has of eventually winning when he is 2 steps away. But $p_2 = p_1^2$ because it is the product of the probability he reaches 1 step away from 2 steps away, and then reaches 0 steps from 1 step. Thus, we obtain an equation

$$p_1 = \frac{1}{4} + \frac{3}{4}p_1^2.$$

Solving for p_1, we obtain the solutions 1 and $\frac{1}{3}$, and the answer that makes sense is $\boxed{\frac{1}{3}}$.