1. It's currently 6:00 on a 12 hour clock. What time will be shown on the clock 100 hours from now? Express your answer in the form $hh : mm$.

   Answer: 10:00

   Solution: We note that adding any multiple of 12 hours does not change the time on the clock. We see that 100 divided by 12 has remainder 4. So that means the time 100 hours from now is equal to the time 4 hours from now, which is $\boxed{10{:}00}$.

2. A tub originally contains 10 gallons of water. Alex adds some water, increasing the amount of water by 20%. Barbara, unhappy with Alex's decision, decides to remove 20% of the water currently in the tub. How much water, in gallons, is left in the tub? Express your answer as an exact decimal.

   Answer: 9.6

   Solution: In order to increase the water level by 20%, we simply multiply by 1.2, to obtain 12 gallons. Now, if we remove 20% of the water level, then we are only left with 80% or 0.8 times as much water. So our final answer is $0.8 \cdot 12 = \boxed{9.6}$.

3. There are 2000 math students and 4000 CS students at Berkeley. If 5580 students are either math students or CS students, then how many of them are studying both math and CS?

   Answer: 420

   Solution: We know that the total number of math or CS majors is 5580. However, the total sum of them individually is 6000. Therefore, exactly $6000 - 5580 = \boxed{420}$ students must be both math and CS majors.

4. Determine the smallest integer x greater than 1 such that x^2 is one more than a multiple of 7.

   Answer: 6

   Solution:

   An easy way to do this problem is to consider x modulo 7. Since 7 is prime, you know that $x^2 - 1 \mod 7$ means that $x = 1$ or $-1 \mod 7$. So the smallest number is $\boxed{6}$. Alternatively, you could just try out values until you see that 6 works.

5. Find two positive integers x, y greater than 1 whose product equals the following sum:

   $$9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 + 27 + 29.$$

   Express your answer as an ordered pair (x, y) with $x \leq y$.

   Answer: $(11, 19)$

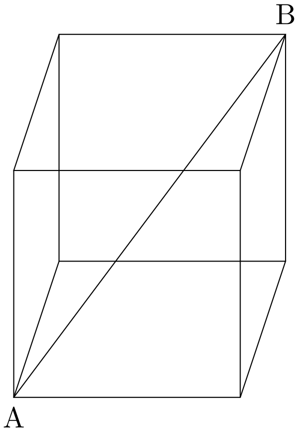
   Solution: This summation is an arithmetic series which can be summed by adding the first and last element and multiplying it by the total number of elements divided by 2. This is equal to $11 \cdot \frac{(9+29)}{2} = 11 \cdot 19$. So $\boxed{(11, 19)}$ is our answer.

6. The average walking speed of a cow is 5 meters per hour. If it takes the cow an entire day to walk around the edges of a perfect square, then determine the area (in square meters) of this square.

   Answer: 900

   Solution: If the cow walks 5 meters per hour, then in one day, the cow walks exactly $5 \cdot 24 = 120$ meters in one day. Therefore, one side of the square is exactly 30 meters, meaning that the area of the square is exactly $30^2 = \boxed{900}$.

7. Consider the cube below. If the length of the diagonal AB is $3\sqrt{3}$, determine the volume of the cube.



Answer: 27

Solution: Let us assume the cube has side length x. Draw the diagonal in the bottom face of the cube. That diagonal has length $x\sqrt{2}$ by the Pythagorean theorem. Additionally this diagonal plus the height from the base to B is x. Therefore, by the Pythagorean theorem again, the length is $x\sqrt{3}$. This means that $x = 3$ so the cube has side length 3. Since the volume of a cube is its side length cubed, we obtain our final answer of $\boxed{27}$.

8. I have 18 socks in my drawer, 6 colored red, 8 colored blue and 4 colored green. If I close my eyes and grab a bunch of socks, how many socks must I grab to guarantee there will be two pairs of matching socks?

Answer: 6

Solution: Given any 4 socks, I must have a pair since by pigeonhole principle, one of them must have a repeated value. Then, by taking that pair out, we have two socks. Adding two more socks to this yields another set of 4 socks, which also must have a pair. So the total number of socks is $4 + 2 = \boxed{6}$. Note that we can have 5 socks without having two pairs if we have 3 red ones, 1 green one, and 1 blue one.

9. Define the operation $a@b$ to be $3 + ab + a + 2b$. There exists a number x such that $x@b = 1$ for all b. Find x.

Answer: −2

Solution: We have $3+xb+x+2b = 1$ so $xb+x+2b+2 = 0$ which means that $(x+2)(b+1) = 0$. We see that if $x = -2$, then this expression is always true. Indeed, plugging in -2 for x yields $3 - 2b - 2 + 2b = 1$. So our answer is $\boxed{-2}$.

10. Compute the units digit of $2017^{(2017^2)}$.

Answer: 7

Solution: Find the units digit of 2017 to the power of 1, 2, etc. until we find a repeating pattern. The units digit of 2017^1 is 7, the units digit of 2017^2 is 9, 2017^3 is 3, 2017^4 is 1, and 2017^5 is 7 again. We see that the units digit of the powers of 2017 cycle every 4. If the power of 2017 has a remainder of 1 when divided by 4, the number will have a units digit of 7. If the remainder is 2, the units digit will be 9, etc. The remainder of $2017^2 = 4068289$ divided by 4 is 1. Therefore, the units digit of 2017^{2017^2} is $\boxed{7}$.

11. The distinct rational numbers $-\sqrt{-x}$, x, and $-x$ form an arithmetic sequence in that order. Determine the value of x.

Answer: $-\frac{1}{9}$

Solution: The key to this problem is understanding that between any two consecutive numbers in an arithmetic sequence, there exists a constant common difference. This means that $-x-x = x+\sqrt{-x}$. Combining the x terms, we have $3x = -\sqrt{-x}$. This means that $9x^2 = -x$. We see that x cannot be zero, because otherwise the terms in the arithmetic sequence would all be zero, and consequently not distinct. Therefore, we may divide x out of the equation obtaining $9x = -1$. Therefore, $x = \boxed{-\frac{1}{9}}$.

12. Let $y = x^2 + bx + c$ be a quadratic function that has only one root. If b is positive, find $\frac{b+2}{\sqrt{c}+1}$.

Answer: 2

Solution: Since it has only one root, we know that the discriminant $b^2 - 4ac$ is 0. Since $a = 1$, we have $b^2 = 4c$. Therefore $\frac{b+2}{\sqrt{c}+1} = \frac{2\sqrt{c}+2}{\sqrt{c}+1} = \boxed{2}$.

13. Alice, Bob, and four other people sit themselves around a circular table. What is the probability that Alice does not sit to the left or right of Bob?

Answer: $\frac{3}{5}$

Solution:

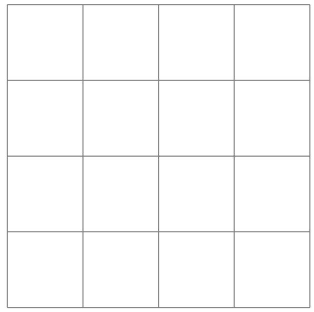
Seat Bob in any of the six places. Then, we have exactly 3 places to sit Alice such that she is not next to Bob. Finally, there are 4! ways to arrange the rest of the people. Therefore the total number of desired arrangements is $6 \cdot 4! \cdot 3 = 18 \cdot 24 = 432$. The total number of arrangements is $6! = 720$. Therefore, the probablity that Alice does not sit to the left or right of Bob is $\frac{432}{720} = \boxed{\frac{3}{5}}$.

14. Let $f(x) = |x-8|$. Let p be the sum of all the values of x such that $f(f(f(x))) = 2$ and q be the minimum solution to $f(f(f(x))) = 2$. Compute $p \cdot q$.

Answer: -480

Solution: Writing out $f(f(f(x)))$ we have $f(f(f(x))) = |||x-8|-8|-8| = 2$. So we have two cases, either $||x-8|-8|-8 = 2$ or $||x-8|-8|-8 = -2$. In this case, we have $||x-8|-8| = 10$ and $||x-8|-8| = 6$. This yields four cases. In particular, $|x-8|-8 = 10$, $|x-8|-8 = -10$, $|x-8|-8 = 6$ and $|x-8|-8 = -6$. Simplifying, we obtain $|x-8| = 18$, $|x-8| = -2$, $|x-8| = 14$, and $|x-8| = 2$. We can discard the second case entirely, since absolute values must be positive. This yields a final 6 cases, which are $x-8 = 18$, $x-8 = -18$, $x-8 = 14$, $x-8 = -14$, $x-8 = 2$ and $x-8 = -2$. Solving we get six solutions: 26, 22, -10, -6, 10 and 6 for a final answer of $48 \cdot -10 = \boxed{-480}$.

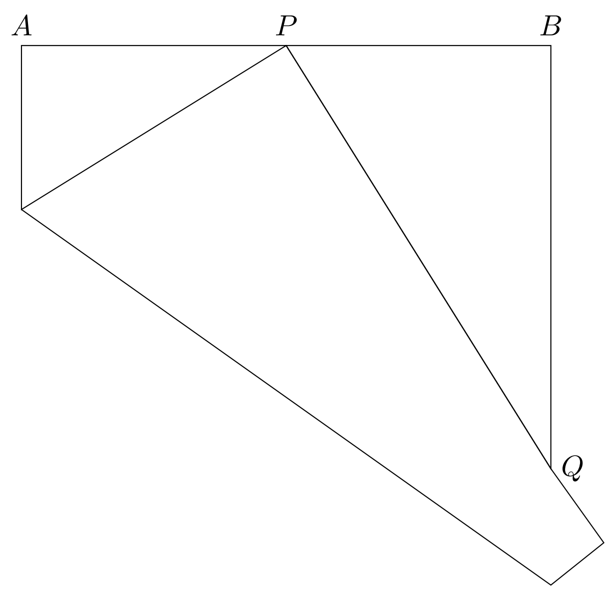
15. Determine the total number of rectangles (1×1, 1×2, 2×2, etc.) formed by the lines in the figure below:



Answer: 100

Solution: A rectangle is made up of choosing 2 horizontal lines and 2 vertical lines. There are $\binom{5}{2} = 10$ ways to choose the horizontal lines and $\binom{5}{2} = 10$ ways to choose the vertical ones. Multiplying, we get the total number of rectangles, which is $\boxed{100}$.

16. Take a square $ABCD$ of side length 1, and let P be the midpoint of AB. Fold the square so that point D touches P, and let the intersection of the bottom edge DC with the right edge be Q. What is BQ?



Answer: $\frac{2}{3}$

Solution: Let R be the point of intersection on the left edge, and let $x = AR$. Then $RP = 1 - x$ and the Pythagorean Theorem tells us

$$x^2 + \frac{1}{4} = (1 - x)^2.$$

This has a solution $x = \frac{3}{8}$. Then

$$\frac{\frac{1}{2}}{\frac{3}{8}} = \frac{BQ}{\frac{1}{2}},$$

so $BQ = \boxed{\frac{2}{3}}$.

17. Let A, B, and k be integers, where k is positive and the greatest common divisor of A, B, and k is 1. Define $x\#y$ by the formula $x\#y = \frac{Ax+By}{kxy}$. If $8\#4 = \frac{1}{2}$ and $3\#1 = \frac{13}{6}$, determine the sum $A + B + k$.

    Answer: 5

    Solution: Let $a = \frac{A}{k}$ and $b = \frac{B}{k}$. Then the operation simplifies down to $\frac{ax+by}{xy}$. Plugging in the given values for the operation, we obtain the two equations $8a + 4b = 16$ and $3a + b = \frac{13}{2}$. Simplifying the equations we obtain $2a + b = 4$ and $6a + 2b = 13$. Subtracting twice the second equation from the first, we get $2a = 5$ and $a = \frac{5}{2}$. Plugging into the first equation, we see that $b = -1$. Now we want to convert back from to A, B, and k. We see that k must be 2, so $A = 5$ and $B = -2$. Therefore our answer is $A + B + k = \boxed{5}$.

18. There are 20 indistinguishable balls to be placed into bins A, B, C, D, and E. Each bin must have at least 2 balls inside of it. How many ways can the balls be placed into the bins, if each ball must be placed in a bin?

    Answer: 1001

    Solution: If each bin has at least two balls inside of it, then we only need to place 10 balls amongst 5 bins. This is a typical stars and bars problem. To solve this we will actually take one ball out of each bin so now we have 15 balls. Then placing them in a line, we need to put 4 bars such that they divide the 15 balls into groupings. There are 14 places to put each bar. So, our answer is $\binom{14}{4} = \boxed{1001}$ ways.

19. Let T_i be a sequence of equilateral triangles such that

    (a) T_1 is an equilateral triangle with side length 1.

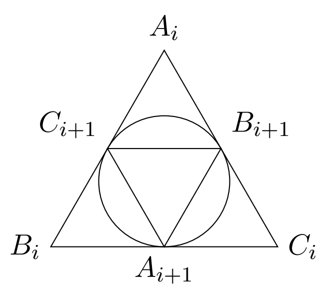
    (b) T_{i+1} is inscribed in the circle inscribed in triangle T_i for $i \geq 1$.

    Find

$$\sum_{i=1}^{\infty} \text{Area}(T_i).$$

Answer: $\frac{\sqrt{3}}{3}$

Let T_i have vertices $A_iB_iC_i$. Note that the area of an equilateral triangle of side length s is $\frac{s^2\sqrt{3}}{4}$. Let s_i be the side length of triangle $A_iB_iC_i$. Then note that $s_{i+1} = \frac{s_i}{2}$. To prove this, we can rotate $A_{i+1}B_{i+1}C_{i+1}$ such that A_{i+1} is the midpoint of B_iC_i, B_{i+1} is the midpoint of side A_iC_i, and C_{i+1} is the midpoint of A_iB_i as shown below:



Therefore, Area(T_{i+1}) is $\frac{1}{4}$ the area of Area(T_i). Hence, the sum is

$$\sum_{i=1}^{\infty} \text{Area}(T_i) = \frac{\sqrt{3}}{4}\sum_{i=1}^{\infty} s_i^2 = \frac{\sqrt{3}}{4}\sum_{i=0}^{\infty}\left(\frac{1}{4}\right)^n = \boxed{\frac{\sqrt{3}}{3}}$$

by the geometric series formula.

20. A *gorgeous* sequence is a sequence of 1's and 0's such that there are no consecutive 1's. For instance, the set of all *gorgeous* sequences of length 3 is $\{[1,0,0],[1,0,1],[0,1,0],[0,0,1],[0,0,0]\}$. Determine the number of *gorgeous* sequences of length 7.

Answer: 34

Solution: Let S_n be the number of *gorgeous* sequences of length n. Looking at an arbitrary sequence of length n, we see that it can either start with 1 or 0. If it starts with 0, then we can simply take all the sequences of length $n-1$ and append it to the 0. If it starts with 1, then the next value must be 0. After that, we can take all the sequences of length $n-2$ and append it to the $[1,0]$. So $S_n = S_{n-1} + S_{n-2}$. The first two terms are $S_0 = 1$ and $S_1 = 2$, so S_n is actually the translated Fibonacci numbers! So by recursively calculating the Fibonacci numbers, we obtain $S_7 = \boxed{34}$.

1. Consider a 4×4 lattice on the coordinate plane. At (0,0) is Mori's house, and at (4,4) is Mori's workplace. Every morning, Mori goes to work by choosing a path going up and right along the roads on the lattice. Recently, the intersection at $(2, 2)$ was closed. How many ways are there now for Mori to go to work?

   Answer: 34

   Solution: There are $\binom{8}{4} = 70$ ways to go to work before the intersection was closed. $\binom{4}{2} \cdot \binom{4}{2} = 36$ ways pass through that intersection. Now Mori has $70 - 36 = \boxed{34}$ was to go to work.

2. Given two integers, define an operation $*$ such that if a and b are integers, then $a * b$ is an integer. The operation $*$ has the following properties:

   1. $a * a = 0$ for all integers a.
   2. $(ka + b) * a = b * a$ for integers a, b, k.
   3. $0 \leq b * a < a$.
   4. If $0 \leq b < a$, then $b * a = b$.

   Find $2017 * 16$.

   Answer: 1

   $2017 = 16(126) + 1$, so $2017 * 16 = (16(126) + 1) * 16 = 1 * 16 = \boxed{1}$.

3. Let ABC be a triangle with side lengths $AB = 13$, $BC = 14$, $CA = 15$. Let A', B', C', be the midpoints of BC, CA, and AB, respectively. What is the ratio of the area of triangle ABC to the area of triangle $A'B'C'$?

   Answer: 4

   Solution 1: Since A' is the midpoint of BC and B' is the midpoint of AC, by SAS similarity, $A'B'||AB$. Also by similarity, $A'B' : AB = 1 : 2$. Repeating this argument for the other sides, we see that $A'B'C'$ has half the side lengths of ABC, and thus has half the heights as the heights in ABC. Therefore, $A'B'C'$ is $\frac{1}{4}$ the area of ABC.

   Solution 2: Drawing lines, one could *surely* convince themselves that the area of $A'B'C'$ is $\frac{1}{4}$ the area of ABC.

4. In a strange world, each orange has a label, a number from 0 to 10 inclusive, and there are an infinite number of oranges of each label. Oranges with the same label are considered indistinguishable. Sally has 3 boxes, and randomly puts oranges in her boxes such that

   (a) If she puts an orange labelled a in a box (where a is any number from 0 to 10), she cannot put any other oranges labelled a in that box.

   (b) If any two boxes contain an orange that have the same labelling, the third box must also contain an orange with that labelling.

   (c) The three boxes collectively contain all types of oranges (oranges of any label).

   The number of possible ways Sally can put oranges in her 3 boxes is N, which can be written as the product of primes:

   $$p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

   where $p_1 \neq p_2 \neq p_3 \cdots \neq p_k$ and p_i are all primes and e_i are all positive integers. What is the sum $e_1 + e_2 + e_3 + \cdots + e_k$?

Answer: 22

Solution: Consider an orange labelled a. It can either be put in box 1 once, box 2 once, box 3 once, or in all three boxes. Therefore, there are 4 places a label a could go. Thus, there are $4^{11} = 2^{22}$ ways Sally can put oranges in her boxes, so the answer is $\boxed{22}$.

5. Suppose I want to stack 2017 identical boxes. After placing the first box, every subsequent box must either be placed on top of another one or begin a new stack to the right of the rightmost pile. How many different ways can I stack the boxes, if the order I stack them doesn't matter? Express your answer as

$$p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$$

where $p_1, p_2, p_3, \ldots, p_n$ are distinct primes and e_i are all positive integers.

Answer: $\mathbf{2^{2016}}$

Solution: Imagine having all 2017 boxes in a row. Between each pair of boxes, you can choose to either stack the boxes on top of each other, or split them up into two stacks. You have two choices for each of 2016 spaces, so you have a total of $\boxed{2^{2016}}$ different ways to stack the boxes.