1. BmMT is in a week, and we don't have any problems! Let's write 1 on the first day, 2 on the second day, 4 on the third, 8 on the fourth, 16 on the fifth, 32 on the sixth, and 64 on the seventh. After seven days, how many problems will we have written in total?

   Answer: 127

   Solution: We can quickly compute the number of problems either by adding them up one by one, or by computing the sum

$$\sum_{i=0}^{6} 2^i = 2^7 - 1 = \boxed{127}.$$

2. 100 students are taking a ten-point exam. 50 students scored 8 points, 30 students scored 7 points, and the rest scored 9 points. What is the average score for the exam?

   Answer: 7.9

   Solution: We can compute that the average score is

$$\frac{(50 \cdot 8) + (30 \cdot 7) + (20 \cdot 9)}{100} = \boxed{7.9}.$$

3. Rebecca has four pairs of shoes. Rebecca may or may not wear matching shoes. However, she will always use a left-shoe for her left foot and a right-shoe for her right foot. How many ways can Rebecca wear shoes?

   Answer: 16

   Solution: There are four choices of shoes for each foot, so the total number of combinations is $4 \cdot 4 = \boxed{16}$.

4. A council of 111 mathematicians voted on whether to hold their conference in Beijing or Shanghai. The outcome of an initial vote was 70 votes in favor of Beijing, and 41 votes in favor of Shanghai. If the vote were to be held again, what is the minimum number of mathematicians that would have to change their votes in order for Shanghai to win a majority of votes?

   Answer: 15

   Solution: For Shanghai to have a majority, they must have at least 56 votes. At least $56 - 41 = \boxed{15}$ pro-Beijing voters must switch in order to give Shanghai the majority that it needs.

5. What is the area of the triangle bounded by the line $20x + 16y = 160$, the x-axis, and the y-axis?

   Answer: 40

   Solution: The line crosses the x-axis and y-axis at $(8, 0)$ and $(0, 10)$, respectively. This creates a right triangle with legs 8 and 10. The area of this triangle is thus $\frac{1}{2} \times 8 \times 10 = \boxed{40}$.

6. Suppose that 3 runners start running from the start line around a circular 800-meter track and that their speeds are 100, 160, and 200 meters per minute, respectively. How many minutes will they run before all three are next at the start line at the same time?

   Answer: 40

   Solution: We wish to compute x such that $100x \equiv 160x \equiv 200x \pmod 8)00$. Simplifying, we simply wish to solve $5x \equiv 8x \equiv 10x \pmod 4)0$. For $5x \equiv 10x \pmod 4)0$ to be true, we know x

must be divisible by 8. For $5x \equiv 8x$, we know x must also be divisible by 5, so our answer is at least 40. We can see that 40 minutes is indeed a valid solution, and is thus the first time after the start that they are together. More concisely, they come back together after $\text{lcm}(4, 5, 8) = \boxed{40}$ minutes.

7. Brian's lawn is in the shape of a circle, with radius 10 meters. Brian can throw a frisbee up to 50 meters from where he stands. What is the area of the region (in square meters) in which the frisbee can land, if Brian can stand anywhere on his lawn?

   Answer: 3600π

   Solution: If Brian stands at the edge of his lawn, his throw can reach 60 meters away from the center of the lawn, so the region he can reach is a circle with radius 60 meters, which has area $\boxed{3600\pi}$.

8. A seven digit number is called "bad" if exactly four of its digits are 0 and the rest are odd. How many seven digit numbers are bad?

   Answer: 1875

   Solution: There are $\binom{6}{4}$ many places where the zeros can go, and there are 5 options for each of the other three digits. This gives a total of $15 \cdot 5^3 = \boxed{1875}$ numbers.

9. Suppose you have a 3-digit number with only even digits. What is the probability that twice that number also has only even digits?

   Answer: $\frac{9}{50}$

   Solution: None of the digits can be a 6 or an 8 because in that case you would need to carry a 1 to the next place when multiplying by 2, which would make the number odd. Therefore, the first digit must be a 2 or a 4, while the remaining two could be any of 0, 2, or 4. This gives us 18 numbers that work out of $\binom{4}{1} \cdot 5^2 = 100$ that have only even digits, which works out to a probability of $\boxed{\frac{9}{50}}$.

10. You have a flight on Air China from Beijing to New York. The flight will depart any time between 1 p.m. and 6 p.m., uniformly at random. Your friend, Henry, is flying American Airlines, also from Beijing to New York. Henry's flight will depart any time between 3 p.m. and 5 p.m., uniformly at random. What is the probability that Henry's flight departs before your flight?

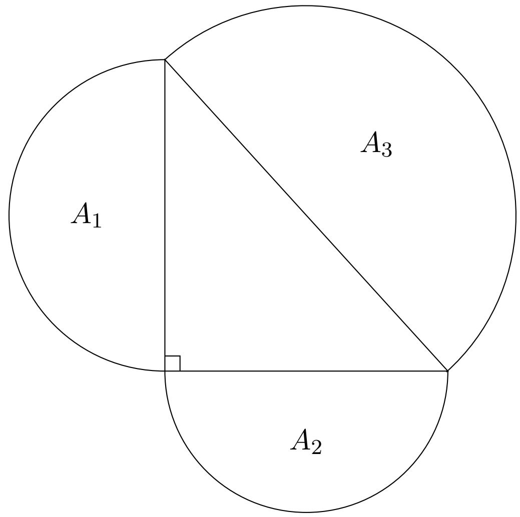
    Answer: $\frac{2}{5}$

    Solution: This is a geometric probability problem. Draw a rectangle with vertices

    $$\{(0, 2), (0, 4), (5, 2), (5, 4)\},$$

    and draw the line $x = y$ through it. The region of the rectangle that lies to the right and below this line represents the event that your friend departs before you do. The ratio of the area of this region to the the area of the entire rectangle, which represents the full probability space, is $\boxed{\frac{2}{5}}$.

11. In the figure below, three semicircles are drawn outside the given right triangle. Given the areas $A_1 = 17$ and $A_2 = 14$, find the area A_3.

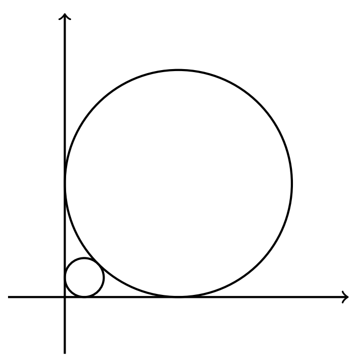


Answer: 31

Solution 1: Let d_1, d_2 and d_3 be the sides of the right triangle. Then by the Pythagorean Theorem, $d_1^2+d_2^2 = d_3^2$. Multiplying both sides by $\pi/8$, we have that $(\pi/8)d_1^2+(\pi/8)d_2^2 = (\pi/8)d_3^2$. In other words, $A_1 + A_2 = A_3$. This gives that $A_3 = 17 + 14 = \boxed{31}$.

Solution 2: Let d_1, d_2 and d_3 be the sides of the right triangle. We know that $A_i = (\pi/8)d_i^2$, and so solving we get that $d_1 = \sqrt{136/\pi}$ and $d_2 = \sqrt{112/\pi}$. Applying the Pythagorean Theorem, we get that $d_3 = \sqrt{248/\pi}$. Now using $A_3 = (\pi/8)d_3^2$, we find that $A_3 = \boxed{31}$.

12. Consider a circle of radius 1 drawn tangent to the positive x and y axes. Now consider another smaller circle tangent to that circle and also tangent to the positive x and y axes. Find the radius of the smaller circle.



Answer: $\frac{\sqrt{2}-1}{\sqrt{2}+1}$ or $3-2\sqrt{2}$

Solution: Note that the distance from the center of the bigger circle to the origin is $\sqrt{2}$ by the Pythagorean Theorem. The distance can also be computed as $r\sqrt{2}+r+1$. Solving for r, we get

$$\boxed{\frac{\sqrt{2}-1}{\sqrt{2}+1}} = \boxed{3-2\sqrt{2}}.$$

13. The following expression is an integer. Find this integer:

$$\frac{\sqrt{20+16\frac{\sqrt{20+16\frac{\sqrt{20+16\dots}}{2}}}{2}}}{2}.$$

Answer: 5

Solution: Let $x = \frac{\sqrt{20+16\frac{\sqrt{20+16\frac{\sqrt{20+16\dots}}{2}}}{2}}}{2}$. From this, substitute x into the expression to get that $x = \frac{\sqrt{20+16x}}{2}$. Solving for x from this expression yields the only positive solution, $x = \boxed{5}$, since square roots are positive.

14. Let $2016 = a_1 \times a_2 \times \cdots \times a_n$ for some positive integers $a_1, a_2, \dots, a_n$. Compute the smallest possible value of $a_1 + a_2 + \cdots + a_n$.

Answer: 23

Solution: Clearly we don't want unnecessary terms $a_i = 1$ since that will increase our sum. So we can assume all $a_i \geq 2$. Now consider the prime factorization of $2016 = 2\times2\times2\times2\times2\times3\times3\times7$. We claim that this is minimal. In particular, if we multiply two or more of the factors, we will end up with a larger value of $a_1 + a_2 + \cdots + a_n$ (since $xy \geq x + y$ for all $x, y \geq 2$). Therefore the answer is simply $2 \cdot 5 + 3 \cdot 2 + 7 = \boxed{23}$.

15. The tetranacci numbers are defined by the recurrence $T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}$ and $T_0 = T_1 = T_2 = 0$ and $T_3 = 1$. Given that $T_9 = 29$ and $T_{14} = 773$, calculate T_{15}.

Answer: 1490

Solution 1: From the definition, we can get the simpler formula $T_n = 2T_{n-1} - T_{n-5}$. Therefore we have $T_{15} = 2T_{14} - T_{10} = 2T_{14} - (2T_9 - T_5)$. Now we've reduced the problem to calculating T_5, which we can quickly calculate $T_5 = 2$. Therefore $T_{15} = \boxed{1490}$.

Solution 2: Using the definition, we calculate 0, 0, 0, 1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, $\boxed{1490}$.

16. Find the number of zeros at the end of $(2016!)^{2016}$. Your answer should be an integer, not its prime factorization.

Answer: 1012032

Solution: To count the number of zeros that are at the end of 2016!, we simply compute the number of 5's in its factorization. To do this, we add the number of multiples of 5 less than or equal to 2016, the number of multiples of 25, etc. This is simply

$$\left\lfloor \frac{2016}{5} \right\rfloor + \left\lfloor \frac{2016}{25} \right\rfloor + \left\lfloor \frac{2016}{125} \right\rfloor + \left\lfloor \frac{2016}{625} \right\rfloor = 403 + 80 + 16 + 3 = 502.$$

Raising 2016! to the power of 2016, the number of zeros would simply be multiplied by 2016, so our answer is $\boxed{1012032}$.

17. A DJ has 7 songs named 1, 2, 3, 4, 5, 6, and 7. He decides that no two even-numbered songs can be played one after the other. In how many different orders can the DJ play the 7 songs?

Answer: 1440

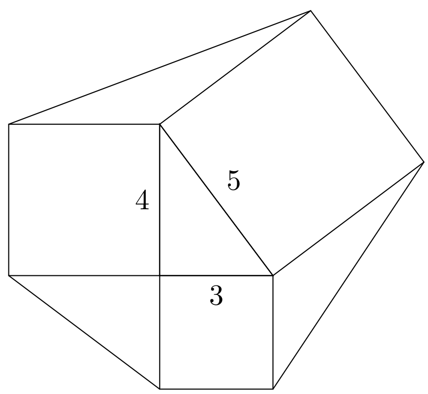
Solution: Start by placing the 3 songs with even numbers. We can use the stars and bars argument (place the 3 songs, then place 2 slots in between the pairs, then place 2 slots among 4 "boxes") to show that there are $\binom{2+4-1}{2} = 10$ ways to place them. Therefore, there are 3! ways to order the even songs and 4! ways to order the odd-numbered songs (which must take the remaining 4 slots). Our answer is $10 \times 6 \times 24 = \boxed{1440}$.

18. Given a cube, how many distinct ways are there (using 6 colors) to color each face a distinct color? Colorings are distinct if they cannot be transformed into one another by a sequence of rotations.

Answer: 30

Solution: For any coloring, we can always orient the cube such that color 1 is on top. We then have 5 different choices for the face opposite to color 1 (the bottom face of the cube). For the faces along the side, this is equivalent to arranging 4 things around a circular table, so there are 6 such distinct colorings. In total, there are thus $5 \cdot 6 = \boxed{30}$ distinct colorings.

19. Suppose you have a triangle with side lengths 3, 4, and 5. For each of the triangle's sides, draw a square on its outside. Connect the adjacent vertices in order, forming 3 new triangles (as in the diagram). What is the area of this convex region?



Answer: 74

Solution: The area of the 3–4–5 triangle is 6, and the areas of the squares are 9, 16, and 25. Between the squares of side length 3 and 4, there will be another 3–4–5 right triangle, which will have area 6. The slope of the non-vertical and non-horizontal line of the 3–4–5 right triangle in the center will be $\frac{3}{4}$, so the diagonals of the square of side length 5 which form the vertices of the other two outside triangles will have slope $-\frac{4}{3}$. We can use this fact to see that the triangle with base 3 will have height 4, and that the triangle with base 4 will have height 3, meaning that both of their areas are 6. Therefore, the total area is $6 + 9 + 16 + 25 + 6 + 6 + 6 = \boxed{74}$.

20. Find x such that $\sqrt{c + \sqrt{c - x}} = x$ when $c = 4$.

Answer: $\mathbf{\frac{1}{2}(1 + \sqrt{13})}$

Solution: Doing some algebra gives that $x^4 - 2cx^2 + x + c^2 - c = 0$. Solving this in terms of x would be annoying, so we will solve in terms of c. Writing $c^2 - (2x^2 + 1)c + (x^4 + x) = 0$ and applying the quadratic formula, we get

$$c = \frac{(2x^2+1) \pm \sqrt{(2x^2+1)^2 - 4(x^4+x)}}{2}$$
$$c = \frac{(2x^2+1) \pm |1-2x|}{2}.$$

Based on the absolute value sign, we have two cases: $x \geq 1/2$ and $x < 1/2$.

If $x \geq 1/2$, our equation reduces to $c = x^2 - x + 1$. Letting $c = 4$, we find that $x = \frac{1}{2}(1 \pm \sqrt{13})$. Since $x \geq 1/2$ we cannot have $x = \frac{1}{2}(1 - \sqrt{13})$.

If $x < 1/2$, then our equation reduces to $c = x^2 + x$. Letting $c = 4$, we find that $x = \frac{1}{2}(-1 \pm \sqrt{17})$. We know that $x \geq 0$ since x is equal to the square root of some number $(c + \sqrt{c-x})$. But now $0 \leq x < 1/2$, and we see that neither of those two solutions are actually possible.

Therefore the only solution is $x = \boxed{\frac{1}{2}(1+\sqrt{13})}$.