1. What is the units digit of $1 + 9 + 9^2 + \ldots + 9^{2015}$?

   Solution: The even powers have units digit 1 and the odd powers have units digit 9, so the sum has units digit 0.

2. In Fourtown, every person must have a car and therefore a license plate. Every license plate must be a 4-digit number where each digit is a value between 0 and 9 inclusive. However 0000 is not a valid license plate. What is the minimum population of Fourtown to guarantee that at least two people who have the same license plate?

   Solution: Since there are 9999 possible license plates, there would need to be $10,000$ people in Fourtown to guarantee a duplicate plate.

3. Two sides of an isosceles triangle $\triangle ABC$ have lengths 9 and 4. What is the area of $\triangle ABC$?

   Solution: The third side must be 9, since a $4, 4, 9$ triangle can't exist. If we drop a perpendicular from the vertex opposite the base to the side of length 4, then we have a right triangle with leg 2 and hypotenuse 9, so the altitude has length $\sqrt{9^2 - 2^2} = \sqrt{77}$. Thus, the area of the triangle is $4\sqrt{77}/2 = 2\sqrt{77}$.

4. Let x be a real number such that $10^{\frac{1}{x}} = x$. Find $(x^3)^{2x}$

   Solution: $(x^3)^{2x} = (x^x)^6 = 10^6 = 1,000,000$.

5. A Berkeley student and a Stanford student are going to visit each others campus and go back to their own campuses immediately after they arrive by riding bikes. Each of them rides at a constant speed. They first meet at a place 17.5 miles away from Berkeley, and secondly 10 miles away from Stanford. How far is Berkeley away from Stanford in miles?

   Solution: If x is the distance between the two schools, s the speed of the Stanford student, and c the speed of the Berkeley student, then the first meeting point gives $17.5/c = (x - 17.5)/s$. The second meeting point gives $(x + 10)/c = (2x - 10)/s$. WLOG, we can assume that $s = 1$, so we get $17.5/c = x - 17.5$ and $(x + 10)/c = 2x - 10$. From this, we get $c = 7/10, x = 85/2$, so Berkeley and Stanford are $85/2 = 42.5$ miles apart.

6. Let $ABCDEF$ be a regular hexagon. Find the number of subsets S of $\{A, B, C, D, E, F\}$ such that every edge of the hexagon has at least one of its endpoints in S.

   Solution: Since each vertex can cover at most 2 edges, S must contain at least 3 vertices. There are two sets of 3 vertices that cover all the edges: $\{A, C, E\}, \{B, D, F\}$. Any set of 4 vertices covers all edges unless the two missing edges are adjacent, giving $\binom{6}{4} - 6 = 9$ sets of 4. Each of the 6 subsets of 5 vertices and the subset of 6 vertices covers all the edges, so the total number of subsets is $2 + 9 + 5 + 1 = 17$.

7. A three digit number is a multiple of 35 and the sum of its digits is 15. Find this number.

   Solution: Since the sum of digits is divisible by 3, the number must be as well, so it's in fact a multiple of $35 \cdot 5 = 105$. Checking the three digit multiples of 105, we see that 735 is the unique solution.

8. Thomas, Olga, Ken, and Edward are playing the card game SAND. Each draws a card from a 52 card deck. What is the probability that each player gets a different rank and a different suit from the others?

   Solution: Thomas has 52 choices, then Olga has $12 \cdot 3 = 36$ choices, then Ken has $11 \cdot 2 = 22$ choices, and finally Edward has $10 \cdot 1 = 10$ choices. Dividing this by the $52(51)50(49)$ possible ways to draw the cards, we get a probability of $\frac{264}{4165}$.

9. An isosceles triangle has two vertices at $(1,4)$ and $(3,6)$. Find the x-coordinate of the third vertex assuming it lies on the x-axis.

   Solution: Since the distance between the two given points is $2\sqrt{2}$, and each point is further away than that from the x axis, the two equal sides must be the sides meeting at the vertex on the x-axis. Thus, we need to find the x such that $(x-1)^2+4^2=(x-3)^2+6^2$. Rearranging, we get $20=(x-1)^2-(x-3)^2=4x-8$, and so $x=7$.

10. Find the number of functions from the set $\{1,2,\dots,8\}$ to itself such that $f(f(x))=x$ for all $1\le x\le 8$.

    Solution: Note that such a function must either fix points or switch pairs of two points. We will do casework on the number of pairs.

    If there are 0 pairs, there is only 1 function, which fixes all points.

    If 1 pair is switched, there are $\binom{8}{2}=28$ ways to choose that pair.

    If 2 pairs are switched, there are $\binom{8}{4}=70$ ways to pick the 4 points to be switched, and 3 ways to switch them in pairs, giving a total of 210.

    If 3 pairs are switched, there are $\binom{8}{6}=28$ ways to pick the 6 points, and $5\cdot 3=15$ ways to pair them, for a total of $28\cdot 15=420$ ways.

    If all 4 pairs are switched, there are $7\cdot 5\cdot 3=105$ ways to pair them , giving a total of 105 ways.

    Adding these all up, we get a grand total of $1+28+210+420+105=764$ possible functions.

11. The circle has the property that, no matter how it's rotated, the distance between the highest and the lowest point is constant. However, surprisingly, the circle is not the only shape with that property. A Reuleaux Triangle, which also has this constant diameter property, is constructed as follows. First, start with an equilateral triangle. Then, between every pair of vertices of the triangle, draw a circular arc whose center is the 3rd vertex of the triangle. Find the ratio between the areas of a Reuleaux Triangle and of a circle whose diameters are equal.

    Solution: Consider a Reuleaux Triangle whose equilateral triangle has side length 1. Then, the radii of each of the three circular arcs is 1, and so the diameter is 1. We can find the area of this shape by tripling the area of a circular sector of radius 1 and angle 60 degrees, and then subtracting twice the area of an equilateral triangle of side length 1. From this, we find an area of $\frac{3}{6}\pi-2\frac{\sqrt{3}}{4}=\frac{\pi}{2}-\frac{\sqrt{3}}{2}$. The area of a circle of diameter 1 is $\frac{\pi}{4}$. Therefore, the desired ratio is $2-\frac{2\sqrt{3}}{\pi}$.

12. Let a,b,c be positive integers such that $\gcd(a,b)=2$, $\gcd(b,c)=3$, $\text{lcm}(a,c)=42$, and $\text{lcm}(a,b)=30$. Find abc.

    Solution: b is divisible by 2 and 3, and b divides 30, so we have $b=6$ or $b=30$. Then $a=10$ or $a=2$, respectively. Since $\text{lcm}(a,c)=42$, we cannot have $a=10$. Then $a=2$, so we have $c=21$ or $c=42$. If $c=42$, then $\gcd(b,c)=6$, which is not true, so we must have $c=21$. This satisfies all our conditions, so we have $abc=(2)(30)(21)=1260$.

13. A point P is inside the square $ABCD$. If $PA=5, PB=1, PD=7$, then what is PC?

    Solution: If we drop perpendiculars from P to each of the four sides of the square, then we can use the Pythagorean Theorem to conclude that $PA+PC$ and $PB+PD$ are both equal to the sum of the lengths of the perpendiculars. Therefore, $5^2+PC^2=1^2+7^2$, and so $PC=5$.

14. Find all positive integers n such that, for every positive integer x relatively prime to n, we have that n divides $x^2 - 1$. You may assume that if $n = 2^k m$, where m is odd, then n has this property if and only if both 2^k and m do.

    Solution: If 2^k has the desired property, then since 3 is relatively prime to 2, we must have $2^k | 3^2 - 1 = 8$, so $k = 0, 1, 2, 3$. Since 2 is relatively prime to m, we also must have $m | 2^2 - 1 = 3$, so $m = 1, 3$. Thus, the possible n are $1, 2, 3, 4, 6, 8, 12, 24$. Conversely, since x is relatively prime to n iff it's relatively prime to both 2^k and m, and since $n | x^2 - 1$ iff $2^k | x^2 - 1$ and $m | x^2 - 1$, all of these numbers satisfy the desired property. Therefore, the answer is $\{1, 2, 3, 4, 6, 8, 12, 24\}$.

15. Given integers a, b, c satisfying

$$abc + a + c = 12$$
$$bc + ac = 8$$
$$b - ac = -2,$$

    what is the value of a?

    Solution: Adding the first and third equations and subtracting the second, we get $2 = abc + a + c - bc - ac + b - ac = (a-1)(b-1)(b-1) + 1$, so $(a-1)(b-1)(b-1) = 1$. Since a, b, c are integers, we have $a - 1 = -1$ or $a - 1 = 1$. If $a - 1 = -1$, we have $a = 0$, which leads to a contradiction. Testing $a - 1 = 1$, we get $a = b = c = 2$, so we have $a = 2$.

16. Two sides of a triangle have lengths 20 and 30. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side?

    Solution: Let the height to the side of length 30 be h_1, the height to the side of length 20 be h_2, the area be A, and the height to the unknown side be h_3.

    Because the area of a triangle is $\frac{bh}{2}$, we get that $30(h_1) = 2A$ and $20(h_2) = 2A$, so, setting them equal, $h_2 = \frac{3h_1}{2}$. From the problem, we know that $2h_3 = h_1 + h_2$. Substituting, we get that

$$h_3 = \frac{5}{4}h_1.$$

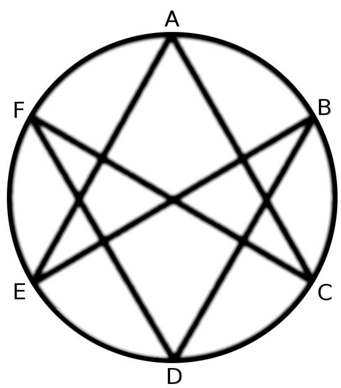
    Thus, the side length is going to be $\frac{2A}{(5/4)h_1} = \frac{30}{5/4} = 24$

17. Find the number of non-negative integer solutions (x, y, z) of the equation

    $xyz + xy + yz + zx + x + y + z = 2014.$

    Solution: Adding 1 to both sides, we get $(x+1)(y+1)(z+1) = 2015$, and thus we seek positive integer solutions of $XYZ = 2015$, where $X = x+1, Y = y+1, Z = z+1$. Since $2015 = 5 \cdot 13 \cdot 31$, there are $3^3 = 27$ ways to put each prime into one of the three factors, and thus there are 27 solutions.

18. Assume that A, B, C, D, E, F are equally spaced on a circle of radius 1, as in the figure below. Find the area of the kite bounded by the lines EA, AC, FC, BE.



Solution: Obviously, one diagonal of the kite has length 1. Half of the other diagonal is the x-coordinate of the intersection point of the lines between $(0,1)$ and $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and the line between $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. These lines are $y = -\sqrt{3}x + 1$ and $y = \frac{x}{\sqrt{3}}$, so their intersection is at $x = \frac{\sqrt{3}}{4}$. Thus, since the area of a kite is half the product of the diagonals, the area is $\frac{\sqrt{3}}{4}$.

19. A positive integer is called cyclic if it is not divisible by the square of any prime, and whenever $p < q$ are primes that divide it, q does not leave a remainder of 1 when divided by p. Compute the number of cyclic numbers less than or equal to 100.

    Solution: Firstly, 1 and all primes are cyclic. This gives 26 cyclic numbers. Next, we consider products of two primes: since all non-2 primes are odd and thus leave a remainder of 1 upon division by 2, both primes must be odd. The combinations that work are $3 \cdot 5, 3 \cdot 11, 3 \cdot 17, 3 \cdot 23, 3 \cdot 29, 5 \cdot 7, 5 \cdot 13, 5 \cdot 17, 5 \cdot 19, 7 \cdot 11, 7 \cdot 13$. Products of three or more primes don't work, since the product of the first three odd primes, $3 \cdot 5 \cdot 7$, is greater than 100. Thus, we have just 11 additional cyclic numbers, for a total of 37.

20. On an 8×8 chess board, a queen can move horizontally, vertically, and diagonally in any direction for as many squares as she wishes. Find the average (over all 64 possible positions of the queen) of the number of squares the queen can reach from a particular square (do not count the square she stands on).

    Solution: By symmetry, we only need to consider the average over the 16 squares in a quadrant. Clearly, the queen can reach 14 squares in horizontal and vertical directions, no matter where she is.

    If the queen is on an edge square(including a corner), the number of squares she can reach diagonally is 7. If the queen is 1 square from an edge square, she can reach 9 squares diagonally. She can diagonally reach 11 squares from a square 2 away from the nearest edge square, and she can reach 13 squares from the one square in the quadrant that's 3 spaces away from the nearest edge square. Since there are 7 edge squares, 5 squares 1 away from an edge square, 3 squares 2 away, and 1 square 1 away, this gives an average diagonal power of $\frac{1}{16}(7 \cdot 7 + 5 \cdot 9 + 3 \cdot 11 + 1 \cdot 13) = \frac{35}{4}$. Adding in the constant horizontal/vertical power of 14, we get a total average power of $\frac{91}{4}$.