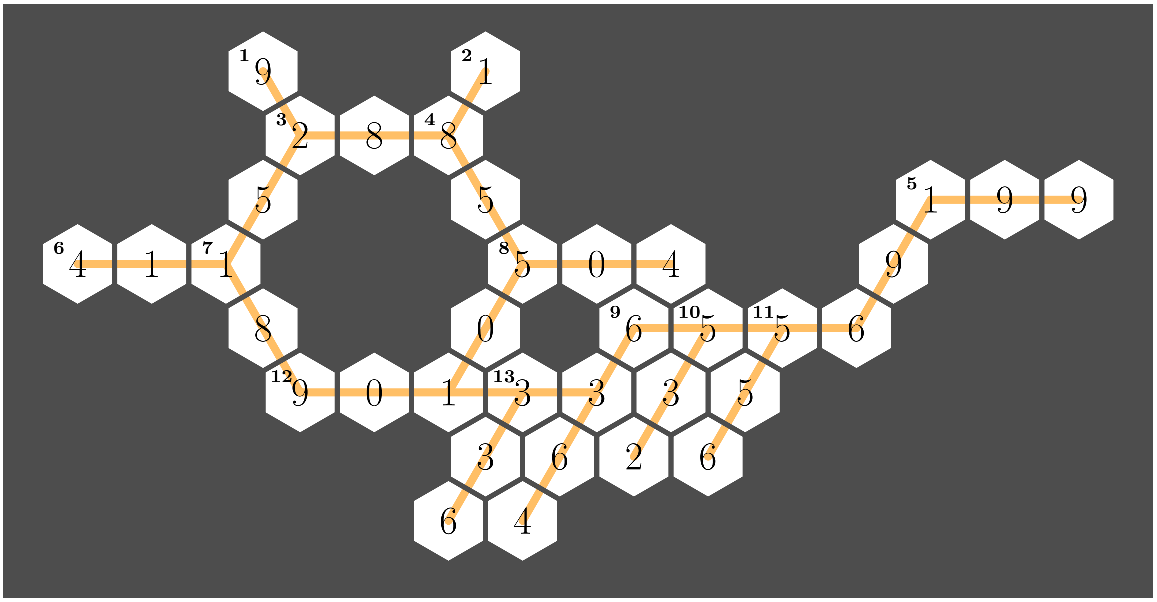
PUMaC 2012

Team Round Solutions

1 Instructions



- **Time limit: 20 minutes.**
- Fill in the crossword above with answers to the problems below.
- Notice that there are three directions instead of two. You are probably used to "down" and "across," but this crossword has "1," "$e^{4\pi i/3}$," and "$e^{5\pi i/3}$." You can think of these labels as complex numbers pointing in the direction to fill in the spaces. In other words "1" means "across," "$e^{4\pi i/3}$" means "down and to the left," and "$e^{5\pi i/3}$" means "down and to the right."
- To fill in the answer to, for example, 12 across, start at the hexagon labeled 12, and write the digits, proceeding to the right along the gray line. (Note: 12 across has space for exactly 5 digits.)
- Each hexagon is worth one point, and must be filled by something from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Note that π is not in the set, and neither is i, nor $\sqrt{2}$, nor $\heartsuit$, etc.
- None of the answers will begin with a 0.
- "Concatenate a and b" means to write the digits of a, followed by the digits of b. For example, concatenating 10 and 3 gives 103. (It's not the same as concatenating 3 and 10.)
- Calculators are allowed!
- THIS SHEET IS PROVIDED FOR YOUR REFERENCE ONLY. DO NOT TURN IN THIS SHEET. TURN IN THE OFFICIAL ANSWER SHEET PROVIDED TO THE TEAM. OTHERWISE YOU WILL GET A SCORE OF ZERO! ZERO! ZERO! AND WHILE SOMETIMES "!" MEANS FACTORIAL, IN THIS CASE IT DOES NOT.
- Good luck, and have fun!

2 The problems

2.1 Across (1)

3. **Problem:** (3 digits) Suppose you draw 5 vertices of a convex pentagon (but not the sides!). Let N be the number of ways you can draw at least 0 straight line segments between the vertices so that no two line segments intersect in the interior of the pentagon. What is $N - 64$? (Note what the question is asking for! You have been warned!)

   Answer: 288

   Solution: Note that the five outer edges don't really matter. As in we don't have to worry about whether they intersect anything else.

   Case work! Let the five vertices be A, B, C, D, E in that order. If AC is drawn other interior segments we could have are $\{CE\}$, $\{AD\}$, $\emptyset$, so 3 in this case. If AC is not drawn, we could have $\emptyset$, $\{AD\}$, $\{AD, BD\}$ $\{BD\}$, $\{BE\}$, $\{BD, BE\}$, $\{BE, CE\}$, $\{CE\}$, so 8 in this case. Thus, 11 total for the interior segments.

   Multiply by 2^5 for the outer edges, so there are $11 \times 32 = 352$ ways. The answer is then $352 - 64 = \boxed{288}$.

   Author: Alan

5. **Problem:** (3 digits) Among integers $\{1, 2, \dots, 10^{2012}\}$, let n be the number of numbers for which the sum of the digits is divisible by 5. What are the first three digits (from the left) of n?

   Answer: 199

   Solution: Since 10^{2012} does not work, we only need to concern ourselves with numbers with 2012 digits. Let $n = a_0 + a_1 10 + a_2 10^2 + \cdots + a_{2011} 10^{2011}$. Each digit $a_0, \dots, a_{2011}$ can be chosen from $\{0, 1, \dots, 9\}$.

   We have 10 choices for a_0, 10 choices for a_1, ..., 10 choices for a_{2010}, and finally, 2 choices for a_{2011}. (Given any choices for the first 2011 digits, there are exactly 2 choices for the last digit for which the sum is divisible by 5.)

   However, in the process described above, we might have chosen $a_0 = a_1 = \cdots = a_{2011} = 0$, which is not allowed.

   Thus, there are $2 \cdot 10^{2011} - 1$ numbers that work. The number looks like $199 \dots 99$, so the first three digits are $\boxed{199}$.

   Author: Albert

6. **Problem:** (3 digits) Bob is punished by his math teacher and has to write all perfect squares, one after another. His teacher's blackboard has space for exactly 2012 digits. He can stop when he cannot fit the next perfect square on the board. (At the end, there might be some space left on the board - he does not write only part of the next perfect square.) If n^2 is the largest perfect square he writes, what is n?

   Answer: 411

Solution: Since a positive integer k has exactly $1 + \lfloor \log_{10} k \rfloor$ digits, we want to find the largest integer n such that

$$\sum_{k=1}^{n} \left(1 + \lfloor \log_{10} k^2 \rfloor\right) \leq 2012$$

Using the approximation $\sqrt{10} \approx 3.16228\cdots$, we have

$$1 + \lfloor \log_{10} k^2 \rfloor = \begin{cases} 1, & \text{if } k = 1, 2, 3 \\ 2, & \text{if } k = 4, 5, 6, 7, 8, 9 \\ 3, & \text{if } k = 10, 11, \ldots, 31 \\ 4, & \text{if } k = 32, 33, \ldots, 99 \\ 5, & \text{if } k = 100, 101, \ldots, 316 \\ 6, & \text{if } k = 317, 318, \ldots, 999 \end{cases}$$

By the time Bob finishes writing 316^2 on the blackboard, the number of digits he will have written down will be

$$1 \cdot 3 + 2 \cdot 6 + 3 \cdot 22 + 4 \cdot 68 + 5 \cdot 217 = 1438$$

The numbers $317^2, \ldots, 999^2$ all have 6 digits, so he will have space for $\lfloor \frac{2012-1438}{6} \rfloor = 95$ more numbers. The last square he writes is then 411^2.

Author: Alan

8. **Problem:** (3 digits) How many positive integers n are there such that $n \leq 2012$, and the greatest common divisor of n and 2012 is a prime number?

   Answer: 504

   Solution: We know that $2012 = 2^2 \cdot 503$, so $\gcd(n, 2012)$ must be 2 or 503.

   If the gcd is 2, then $n = 2a$, where a can be any number in $\{1, 3, 5, \cdots, 1005\}$ except 503. There are 502 possible values of a. If the gcd is 503, then $n = 503$ or $3 \cdot 503$. Thus, the final answer is $502 + 2 = \boxed{504}$.

   Author: Alan

9. **Problem:** (4 digits) I have a random number machine generator that is very good at generating integers between 1 and 256, inclusive, with equal probability. However, right now, I want to produce a random number between 1 and n, inclusive, so I do the following:

   - I use my machine to generate a number between 1 and 256. Call this a.
   - I take a and divide it by n to get remainder r. If $r \neq 0$, then I record r as the randomly generated number. If $r = 0$, then I record n instead.

   Note that this process does not necessarily produce all numbers with equal probability, but that is okay. I apply this process twice to generate two numbers randomly between 1 and 10. Let p be the probability that the two numbers are equal. What is $p \cdot 2^{16}$?

Answer: 6556

Solution: The probability distribution for each number is:

$$P(k) = \begin{cases} \frac{26}{256}, & \text{if } k \in \{1,2,3,4,5,6\} \\ \frac{25}{256}, & \text{if } k \in \{7,8,9,10\} \end{cases}$$

So the probability is

$$p = \frac{6 \cdot 26^2 + 4 \cdot 25^2}{256^2} = \frac{6556}{256^2}$$

and the answer is $\boxed{6556}$.

Author: Alan

12. **Problem:** (5 digits) You and your friend play the following dangerous game. You two start off at some point (x, y) on the plane, where x and y are nonnegative integers.

When it is player A's turn, A tells his opponent B to move to another point on the plane. Then A waits for a while. If B is not eaten by a tiger, then A moves to that point as well.

From a point (x, y) there are three places A can tell B to walk to: leftwards to $(x-1, y)$, downwards to $(x, y-1)$, and simultaneously downwards and leftwards to $(x-1, y-1)$. However, you cannot move to a point with a negative coordinate.

Now, what was this about being eaten by a tiger? There is a tiger at the origin, which will eat the first person that goes there! Needless to say, you lose if you are eaten.

Consider all possible starting points (x, y) with $0 \le x \le 346$ and $0 \le y \le 346$, and x and y are not both zero. Also suppose that you two play strategically, and you go first (i.e., by telling your friend where to go). For how many of the starting points do you win?

Answer: 90133

Solution: Let's include the origin as a starting point since it will not affect the count.

If we work backwards starting from the origin, we see that you will win if and only if at least one of coordinates the starting point (x, y) is odd. The starting points where you would lose are

$$\{(2a, 2b) : 0 \le a \le 174, 0 \le b \le 174\}$$

Thus, there are 174^2 bad points, so $347^2 - 174^2 = \boxed{90133}$ good points.

Author: Alan

2.2 Down and to the left $(e^{4\pi i/3})$

2. **Problem:** (2 digits) $ABCDE$ is a pentagon with $AB = BC = CD = \sqrt{2}$, $\angle ABC = \angle BCD = 120$ degrees, and $\angle BAE = \angle CDE = 105$ degrees. Find the area of triangle $\triangle BDE$. Your answer in its simplest form can be written as $\frac{a+\sqrt{b}}{c}$, where where a, b, c are integers and b is square-free. Find abc.

Answer: 18

Solution: Subtract the areas of triangles EAB and CDE from the area of the pentagon. The answer should be $(3+\sqrt{3})/2$, so $abc = \boxed{18}$.

Author: Ante

3. **Problem:** (3 digits) Suppose x and y are integers which satisfy

$$\frac{4x^2}{y^2} + \frac{25y^2}{x^2} = \frac{10055}{x^2} + \frac{4022}{y^2} + \frac{2012}{x^2y^2} - 20$$

What is the maximum possible value of $xy - 1$?

Answer: 251

Solution: If we move all the terms to the left and clear the denominators, we have the equation

$$4x^4 + 20x^2y^2 + 25y^4 - 4022x^2 - 10055y^2 - 2012 = 0$$

The left hand side can be simplified as follows:

$$\begin{aligned} &4x^4 + 20x^2y^2 + 25y^4 - 4022x^2 - 10055y^2 - 2012 \\ &= (2x^2 + 5y^2)^2 - 2011(2x^2 + 5y^2) - 2012 \\ &= (2x^2 + 5y^2 - 2012)(2x^2 + 5y^2 + 1) \end{aligned}$$

Thus, we must have $2x^2 + 5y^2 - 2012 = 0$ or $2x^2 + 5y^2 + 1 = 0$. The equation $2x^2 + 5y^2 + 1 = 0$ is boring and has no real solutions, so we look at $2x^2 + 5y^2 - 2012 = 0$. It turns out this has exactly one integer solution: $(x, y) = (14, 18)$. Thus, the answer is $14 \cdot 18 - 1 = \boxed{251}$.

Author: Alan

5. **Problem:** (3 digits) Find the area of the set of all points in the plane such that there exists a square centered around the point and having the following properties:

- The square has side length $7\sqrt{2}$.
- The boundary of the square intersects the graph of $xy = 0$ at at least 3 points.

Answer: 196

Solution: The graph of $xy = 0$ is the union of the lines $x = 0$ and $y = 0$. Since the square can only intersect twice with each line, we are looking for squares that can intersect with both lines at the same time. The maximum area can be covered by turning the square 45 degrees, having one corner barely cover one line, moving up and down that line. Then we can do the same for the other side of the same line, and then for the other line. The center sweeps an area inside a square of side length $7\sqrt{2} \cdot \sqrt{2}$, so the total area is $\boxed{196}$.

Author: Ante

8. **Problem:** (3 digits) Princeton Tiger has a mom that likes yelling out math problems. One day, the following exchange between Princeton and his mom occurred:

- Mom: Tell me the number of zeros at the end of 2012!

- PT: Huh? 2012 ends in 2, so there aren't any zeros.
- Mom: No, the exclamation point at the end was not to signify me yelling. I was not asking about 2012, I was asking about 2012!.

What is the correct answer?

Answer: 501

Solution: We want to find the highest power of 10 which divides $2012! = 2012 \cdot 2011 \cdot 2010 \cdots 2 \cdot 1$. If $10^n | 2012!$, then $2^n | 2012!$ and $5^n | 2012!$. 2012! clearly contains more powers of 2 than powers of 5, so we want to find the largest integer n such that $5^n | 2012!$. Writing $2012! = 2012 \cdot 2011 \cdot 2010 \cdots 2 \cdot 1$, we can take a factor of 5 from each of $5, 10, 15, \dots, 2010$. Then we can take another factor of 5 from $25, 50, 75, \dots, 2000$. We continue this process, to give us $n = \lfloor \frac{2012}{5} \rfloor + \lfloor \frac{2012}{25} \rfloor + \lfloor \frac{2012}{125} \rfloor + \lfloor \frac{2012}{625} \rfloor = \boxed{501}$

Author: Alan

9. **Problem:** (4 digits) Define the following:

- $A = \sum_{n=1}^{\infty} \frac{1}{n^6}$
- $B = \sum_{n=1}^{\infty} \frac{1}{n^6+1}$
- $C = \sum_{n=1}^{\infty} \frac{1}{(n+1)^6}$
- $D = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$
- $E = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^6}$

Consider the ratios $\frac{B}{A}, \frac{C}{A}, \frac{D}{A}, \frac{E}{A}$. Exactly one of the four is a rational number. Let that number be r/s, where r and s are nonnegative integers and $\gcd(r, s) = 1$. Concatenate r, s.

(It might be helpful to know that $A = \frac{\pi^6}{945}$.)

Answer: 6364

Solution: We want to see if we can rewrite the sums of B, C, D, E in terms of A somehow. It's not really obvious what to do with B, so we skip that one. With C, we have:

$$C = \sum_{n=1}^{\infty} \frac{1}{(n+1)^6} = \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots = \left(\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots \right) - 1 = A - 1$$

So $\frac{C}{A} = \frac{A-1}{A} = 1 - \frac{1}{A}$. Since A is not rational, neither is $\frac{C}{A}$. For D, we have:

$$D = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \cdots$$

These are the odd terms in the sum for A. If we try to separate the odd terms from the even terms in A, we get:

$$A = \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \ldots = \left(\frac{1}{1^6} + \frac{1}{3^6} + \ldots \right) + \left(\frac{1}{2^6} + \frac{1}{4^6} + \ldots \right)$$

Note that:

$$\frac{1}{2^6} + \frac{1}{4^6} + \ldots = \frac{1}{2^6}\left(\frac{1}{1^6} + \frac{1}{2^6} + \ldots\right) = \frac{1}{2^6}A$$

Thus, we have $A = D + \frac{1}{2^6}A$, which gives $\frac{D}{A} = \frac{63}{64}$, which is rational. Thus, the answer is $\boxed{6364}$.

Note that $E = D - 1$, so $\frac{E}{A}$ is not rational.

Author: Alan

10. **Problem:** (3 digits) You have a sheet of paper, which you lay on the xy plane so that its vertices are at $(-1, 0), (1, 0), (1, 100), (-1, 100)$. You remove a section of the bottom of the paper by cutting along the function $y = f(x)$, where f satisfies $f(1) = f(-1) = 0$. (In other words, you keep the bottom two vertices.)

    You do this again with another sheet of paper. Then you roll both of them into identical cylinders, and you realize that you can attach them to form an L-shaped elbow tube.

    We can write $f(\frac{1}{3}) + f(\frac{1}{6}) = \frac{a+\sqrt{b}}{\pi c}$, where a, b, c are integers and b is square-free. Find $a+b+c$.

    Answer: 532

    Solution: The function is $f(x) = \frac{1}{\pi}(1 + \cos(\pi x))$, so

    $$f(\tfrac{1}{3}) + f(\tfrac{1}{6}) = \frac{5 + \sqrt{3}}{\pi 2}$$

    and the answer is $\boxed{532}$.

    Author: Henry

11. **Problem:** (3 digits) Let

    $$\Xi(x) = 2012(x-2)^2 + 278(x-2)\sqrt{2012 + e^{x^2-4x+4}} + 139^2 + (x^2 - 4x + 4)e^{x^2-4x+4}$$

    find the area of the region in the xy-plane satisfying:

    $$\{x \geq 0 \quad \text{and} \quad x \leq 4 \quad \text{and} \quad y \geq 0 \quad \text{and} \quad y \leq \sqrt{\Xi(x)}\}$$

    Answer: 556

    Solution: Let $f(x) = \sqrt{\Xi(x)}$ and observe that $f(x) = 139 + (x-2)\sqrt{2012 + e^{x^2-4x+4}}$. Let S be the set of points (x, y) which satisfy all four inequalities. Let $g(x) = x\sqrt{2012 + e^{x^2}}$. Notice that this function is odd. Thus, $f(x) = g(x-2) + 139$ has $180°$ rotational symmetry around the point $(x, y) = (2, 139)$.

    If we rotate the region S by $180°$ around this $(2, 139)$, we get a new set S', with the following properties: (1) S and S' are disjoint (except at the boundary). (2) The union $S \cup S'$ forms a rectangle with base 4 and height $f(0) + f(4) = 139 + g(-2) + 139 + g(2) = 2 \cdot 139$.

    Thus, the area of S is $\frac{1}{2} \cdot 4 \cdot 2 \cdot 139 = \boxed{556}$.

    Author: Alan

13. **Problem:** (3 digits) Three cones have bases on the same plane, externally tangent to each other. The cones all face the same direction. Two of the cones have radii of 2, and the other cone has a radius of 3. The two cones with radii 2 have height 4, and the other cone has height 6. Let V be the volume of the tetrahedron with three of its vertices as the three vertices of the cones and the fourth vertex as the center of the base of the cone with height 6. Find V^2.

    Answer: 336

    Solution: Note that the triangle containing the centers of the bases of the three cones is the maximum cross sectional area perpendicular to the side of the tetrahedron from the base to the vertex of the cone with radius 3 and height 6. In fact, we can treat it as 2 separate tetrahedrons split in the middle, with the area of this triangle as the areas of the bases. So the volume of the tetrahedron is

$$V = \frac{1}{3}6A$$

    where A is the area of the triangle. This triangle has side lengths of $5, 5, 4$, so its altitude is $\sqrt{21}$ and its area is $2\sqrt{21}$. So the volume squared is:

$$\begin{aligned} V^2 &= 4A^2 \\ &= 4(4)(21) \\ &= 16 \times 21 = \boxed{336} \end{aligned}$$

    Author: Ante

2.3 Down and to the right ($e^{5\pi i/3}$)

1. **Problem:** (2 digits) For some reason, people in math problems like to paint houses. Alice can paint a house in one hour. Bob can paint a house in six hours. If they work together, it takes them seven hours to paint a house. You might be thinking "What? That's not right!" but I did not make a mistake.

    When Alice and Bob work together, they get distracted very easily and simultaneously send text messages to each other. When they are texting, they are not getting any work done. When they are not texting, they are painting at their normal speeds (as if they were working alone). Carl, the owner of the house decides to check up on their work. He randomly picks a time during the seven hours. The probability that they are texting during that time can be written as r/s, where r and s are integers and $\gcd(r, s) = 1$. What is $r + s$?

    Answer: 92

    Solution: Alice's painting speed is 1 house per hour and Bob's is $1/6$ house per hour. Thus, if they don't text, it would take them $1/(1 + 1/6) = 6/7$ of an hour to paint the house.

    The probability is

$$P = \frac{\text{time spent texting}}{\text{total time}} = \frac{7 - 6/7}{7} = \frac{43}{49}$$

    so the answer is $43 + 49 = \boxed{92}$.

    Author: Alan

4. **Problem:** (3 digits) Let $a_1 = 2 + \sqrt{2}$ and $b_1 = \sqrt{2}$, and for $n \geq 1$, $a_{n+1} = |a_n - b_n|$ and $b_{n+1} = a_n + b_n$. The minimum value of $\frac{a_n^2 + a_n b_n - 6b_n^2}{6b_n^2 - a_n^2}$ can be written in the form $a\sqrt{b} - c$, where a, b, c are integers and b is square-free. Concatenate c, b, a (in that order!).

   Answer: 825

   Solution: First, calculate some terms in the two sequenses. Observe that $(a_2, b_2) = (2, 2 + 2\sqrt{2}), (a_3, b_3) = (2\sqrt{2}, 4 + 2\sqrt{2}), (a_4, b_4) = (4, 4 + 4\sqrt{2})$. Hmm... it looks like for $n \geq 2$, we have $(a_n, b_n) = (\sqrt{2})^n(1, 1 + \sqrt{2})$. In fact, this is true! We can prove it by induction!

   Note that
$$\frac{a_n^2 + a_n b_n - 6b_n^2}{6b_n^2 - a_n^2} = -1 + \frac{a_n/b_n}{6 - (a_n/b_n)^2}$$
   so the expression only depends on the ratio a_n/b_n. We have shown that for $n \geq 2$, we have:
$$\frac{a_n}{b_n} = \frac{(\sqrt{2})^n}{(\sqrt{2})^n(1 + \sqrt{2})} = \frac{1}{1 + \sqrt{2}}$$
   For $n = 1$, we have
$$\frac{a_1}{b_1} = \frac{2 + \sqrt{2}}{\sqrt{2}} = 1 + \sqrt{2}$$
   Since $a_2/b_2 < a_1/b_1$, it follows that
$$\frac{a_2/b_2}{6 - (a_2/b_2)^2} < \frac{a_1/b_1}{6 - (a_1/b_1)^2}$$
   We can see this really quickly because $\frac{r}{6-r^2}$ increases as r increases in the interval $r \in (0, \sqrt{6})$. Thus, the minimum value is
$$-1 + \frac{a_2/b_2}{6 - (a_2/b_2)^2} = 5\sqrt{2} - 8$$
   Thus $a = 5, b = 2, c = 8$, and the answer is $\boxed{825}$

   Author: Alan

7. **Problem:** (3 digits) How many solutions are there to $a^{503} + b^{1006} = c^{2012}$, where a, b, c are integers and $|a|, |b|, |c|$ are all less than 2012?

   Answer: 189

   Solution: Note that the equation can be written as $(a)^{503} + (b^2)^{503} = (c^4)^{503}$. Thanks to Princeton math professor Andrew Wiles, who proved Fermat's last theorem, we know that this equation has no nontrivial solutions. Thus, for any solution (a, b, c), at least one of a, b, c must be zero.

   If $a = 0$, then $b^{1006} = c^{2012}$, so $b = c^2$. Note that $44 < \sqrt{2012} < 45$, so we have the following solutions:
$$S_a = \{(0, c^2, c) \mid c = 0, \pm 1, \pm 2, \ldots, \pm 44\}$$

If $b = 0$, then $a^{503} = c^{2012}$, so $a = c^4$. Note that $6 < \sqrt[4]{2012} < 7$, so we have the following solutions:

$$S_b = \{(c^4, 0, c) \mid c = 0, \pm 1, \pm 2, \ldots, \pm 6\}$$

If $c = 0$, then $a^{503} = -b^{1006}$, so $a = -b^2$, so we have the following solutions:

$$S_c = \{(-b^2, b, 0) \mid b = 0, \pm 1, \pm 2, \ldots, \pm 44\}$$

We have $|S_a| = |S_c| = 89$ and $|S_b| = 13$. (Here, $|S|$ denotes the number of elements in S.) However, note that $(0, 0, 0)$ appears in all three solution sets, so:

$$|S_a \cup S_b \cup S_c| = |S_a| + |S_b| + |S_c| - 2 = 89 + 13 + 89 - 2 = \boxed{189}$$

Author: Alan

– Alan Chang