PUMaC 2012

Number Theory B

1. [**3**] When some number a^2 is written in base b, the result is 144_b. a and b also happen to be integer side lengths of a right triangle. If a and b are both less than 20, find the sum of all possible values of a.

2. [**3**] Let M be the smallest positive multiple of 2012 that has 2012 divisors. Suppose M can be written as

$$\prod_{k=1}^{n} p_k^{a_k}$$

where the p_k's are distinct primes and the a_k's are positive integers. Find

$$\sum_{k=1}^{n} (p_k + a_k).$$

3. [**4**] How many factors of $\left(20^{12}\right)^2$ less than 20^{12} are not factors of 20^{12}?

4. [**4**] Albert has a very large bag of candies and he wants to share all of it with his friends. At first, he splits the candies evenly amongst his 20 friends and himself and he finds that there are five left over. Ante arrives, and they redistribute the candies evenly again. This time, there are three left over. If the bag contains over 500 candies, what is the fewest number of candies the bag can contain?

5. [**5**] How many ways can 2^{2012} be expressed as the sum of four (not necessarily distinct) positive squares?

6. [**6**] Let $f_n(x) = n + x^2$. Evaluate the product

$$\gcd\{f_{2001}(2002), f_{2001}(2003)\} \times \gcd\{f_{2011}(2012), f_{2011}(2013)\} \times \gcd\{f_{2021}(2022), f_{2021}(2023)\},$$

where $\gcd\{x, y\}$ is the greatest common divisor of x and y.

7. [**7**] Find the sum of all possible sums $a + b$ where a and b are nonnegative integers such that $4^a + 2^b + 5$ is a perfect square.

8. [**8**] Let $p_1 = 2012$ and $p_n = 2012^{p_{n-1}}$ for all $n > 1$. Find the largest integer k such that $p_{2012} - p_{2011}$ is divisible by 2011^k.