PUMaC 2012

Individual Finals B

1. Let q be a fixed odd prime. A prime p is said to be *orange* if for every integer a there exists an integer r such that $r^q \equiv a \pmod{p}$. Prove that there are infinitely many orange primes.

2. Let $O_1, O_2, \ldots, O_{2012}$ be 2012 circles in the plane such that no circle intersects or contains any other circle and no two circles have the same radius. For each $1 \leq i < j \leq 2012$, let $P_{i,j}$ denote the point of intersection of the two external tangent lines to O_i and O_j, and let T be the set of all $P_{i,j}$ (so $|T| = \binom{2012}{2} = 2023066$). Suppose there exists a subset $S \subset T$ with $|S| = 2021056$ such that all points in S lie on the same line. Prove that all points in T lie on the same line.

3. Find, with proof, all pairs (x, y) of integers satisfying the equation $3x^2 + 4 = 2y^3$.

Please write complete, concise and clear proofs. Have fun! – PUMaC Problem Writers