



## Combinatorics B Solutions

Written by Andy Zhu

1. By linearity of expectation, the answer is  $2 \times \frac{3+3+4+4+5+6+7+8}{8} = \boxed{10}$ .
2. There are 8 even numbers and 7 odd numbers from 16 to 30. For the sum of three integers to be even, either all three must be even, or two must be odd and the last must be even. There are  $\binom{8}{3} = 7 \cdot 8$  ways to choose the three even numbers, and  $8 \cdot \binom{7}{2} = 7 \cdot 24$  ways to choose the one even and two odd integers. In total, there are  $\binom{15}{3} = 7 \cdot 65$  ways to choose three distinct numbers from 15 numbers. Thus, the probability that the sum is even is  $\frac{7 \cdot 8 + 7 \cdot 24}{7 \cdot 65} = \frac{32}{65}$ , and the answer is  $m + n = \boxed{97}$ .
3. We can do this by casework. Notice that there is exactly one fixed element between consecutive permutations, and there are exactly two permutations for each fixed digit as a fixed element. Without loss of generality, suppose the first permutation is given by (123) (six possibilities), and that the second permutation fixes the third element, and so must be (213) (three possibilities). Without loss of generality again, one may assume that the third permutation fixes the second element (between the first and second elements), and so must be (312) (two possibilities). For the fourth permutation, there are two distinct cases: either the first or third elements are fixed.
  - In the former case, one gets the permutation (213). The two permutations left are (231) and (321) and must occur in that order.
  - In the latter case, one gets the permutation (321). The two permutations left are (231) and (213) and must occur in that order.

Hence, there are  $6 \times 3 \times 2 \times (1 + 1) = \boxed{72}$  possible orderings of these permutations.

An equivalent restatement of the problem is to count the number of Hamiltonian paths of the complete bipartite graph  $K_{3,3}$  (each bipartition corresponding to a copy of  $A_3 \leq S_3$ ), which is readily seen to be  $6 \times 3 \times 2 \times 2 = \boxed{72}$ .

Problem contributed by Luke Paulsen.

4. For each value of  $|S|$  between 1 and  $N - 2$ , the subset  $S$  consists of  $|S|$  of the integers in  $\{1, 2, \dots, |S| + 2\}$ . Thus, there are  $\binom{|S|+2}{|S|}$  possible subsets. Hence the answer is, with  $N = 10$ ,

$$N + 1 + \sum_{i=1}^{N-2} \binom{i+2}{2} = N + \binom{N+1}{3} = \boxed{175},$$

where we applied the hockey-stick identity.

The hockey-stick identity comes with a combinatorial interpretation: instead we can think about choosing 3 distinct integers from  $\{1, 2, \dots, N + 1\}$ : the value of  $|S| + 3$  and the two numbers in  $\{1, 2, \dots, |S| + 2\}$  to exclude from  $S$ .



5. Note that there is some symmetry: the probability of rolling a 9 is the same for both two rolls of the first die and two rolls of the second die (consider the correspondence of a face  $f$  of the first die with that of a face  $9 - f$  on the second die), and is given by  $4/36 = 1/9$  by listing the possible face combinations. If the first and second dice are rolled, the probability of rolling 9 is  $6/36 = 1/6$ . By conditional probability, the answer is

$$P(\text{same dice} \mid \text{rolled 9}) = \frac{P(\text{rolled 9} \mid \text{same dice})}{P(\text{rolled 9})} = \frac{1/9}{\frac{1}{2} \cdot (1/9) + \frac{1}{2} \cdot (1/6)} = \frac{2}{5},$$

so the answer is  $2 + 5 = \boxed{7}$ .

6. Consider the general case for  $P_i = (x_i, y_i) \in \mathbb{Z}_N^2$ , where here  $N = 7$ . For each value of  $M$  with  $0 \leq M \leq N$ , there are  $\binom{N}{M}$  ways to pick each of the  $x$ - and  $y$ -coordinates of the  $M$  points, since the  $M$   $x$ - and  $y$ -coordinates are distinct integers between 0 and  $N$ , inclusive. Hence, the answer is

$$\sum_{M=0}^N \binom{N}{M}^2 = \binom{2N}{N} = \binom{14}{7} = \boxed{3432}$$

by Vandermonde's identity. Alternatively, consider the following bijection to the number of paths from  $(0, -1)$  to  $(N, N-1)$  travelling only up or right at lattice points: keep travelling right until you reach one of the values of  $x_i$ , at which point you move up to  $P_i$  and keep moving right. Conversely, each such path corresponds to the set of points where an 'up' step transitions to a 'right' step.

7. Call the last person at the beginning  $A$ . If  $A$  is at  $i$ th position, for the next round, there is  $\frac{i-1}{5}$  chance that the first person remains in front of  $A$ , which means the position of  $A$  remains the same. Otherwise,  $A$ 's position advances by 1. Hence the expected number of rounds for  $A$  to move the position by 1 is

$$\begin{aligned} \sum_{k=1}^{+\infty} \left(\frac{i-1}{5}\right)^{k-1} \left(1 - \frac{i-1}{5}\right) k &= \sum_{k=0}^{+\infty} \left(\frac{i-1}{5}\right)^k (k+1) - \sum_{k=0}^{+\infty} \left(\frac{i-1}{5}\right)^k k \\ &= \sum_{k=0}^{+\infty} \left(\frac{i-1}{5}\right)^k = \frac{1}{1 - \frac{i-1}{5}} = \frac{5}{5-i+1} \end{aligned}$$

(Note that this is the expectation of a geometric distribution with parameter  $p = \frac{5-i+1}{5}$ .) So in order to move  $A$  to the front, the total expected number of rounds needed is

$$\sum_{i=2}^5 \frac{5}{5-i+1} = \frac{5}{1} + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} = \frac{125}{12},$$

so the answer is  $125 + 12 = \boxed{137}$ .

Problem contributed by Xufan Zhang.

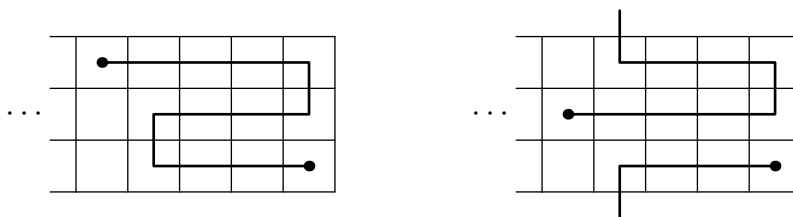
8. Equivalently, consider a Hamiltonian walk on the squares of a cylindrical  $3 \times 40$  chessboard (wrapped so that the two long sides overlap). Let  $A_n$  denote the number of ways to start at the



top left corner  $(1, 3)$  of such a  $3 \times n$  cylindrically-wrapped chessboard and end in the bottom right corner  $(n, 1)$ , and let  $B_n$  denote the number of ways to start at the top left corner  $(1, 3)$  and end in the top right corner  $(n, 3)$ . Then the question is asking for the value of  $B_{40}$ . The key is the two recurrence relations

$$A_n = 1 + \sum_{i=0}^{n-1} A_i + B_i, \quad B_n = 2 \sum_{i=0}^{n-1} A_i \quad (*)$$

This comes from the observation that for any path starting on the left side and ending on the right side, if there were  $k$  horizontal steps before reaching the destination point on the right side, then the previous  $2k + 1$  moves before those are also fixed upto two possibilities:



This holds true for each value of  $k$  from 0 to  $n - 2$ . Note that the number of paths from column 1 to column  $n$  is then reduced to counting the number of paths from column 1 to column  $n - k - 1$ , upto a change in the row. The one exception is when  $k = n - 1$ , in which case we end up with a valid path of the entire  $3 \times n$  board if the destination is not in the first row (hence the  $+1$  in  $(*)$ ). When we keep track of the row change, we obtain the recursions in  $(*)$  - recall the cylindrical nature of the chessboard means that there is symmetry in the cases of the bottom two rows.

It thus suffices to solve  $(*)$ . Notice that

$$A_{n+1} - A_n = A_n + B_n, \quad B_{n+1} - B_n = 2A_n.$$

Substituting the first equation into the second one gives

$$(A_{n+2} - 2A_{n+1}) - (A_{n+1} - 2A_n) = 2A_n \implies A_{n+2} = 3A_{n+1}.$$

Hence, for all  $n > 1$ , one has the explicit formula  $A_n = 3^{n-2}A_2$ . With a bit of casework and visualization, we indeed see that  $A_1 = 1, A_2 = 2, A_3 = 6, \dots$ , so  $A_n = 2 \cdot 3^{n-2}$  for all  $n > 1$ . Thus, the desired answer is the value of  $A_{40} = 2 \cdot 3^{38} \pmod{100}$ . Since  $\varphi(100) = \frac{1}{2} \cdot \frac{4}{5} \cdot 100 = 40$ , Euler's totient theorem shows that  $3^{40} \equiv 1 \pmod{100}$ . Also, we note that  $1 \equiv 201 \equiv 3 \cdot 67 \pmod{100}$ , so

$$A_{40} = 2 \cdot 3^{38} \equiv 2 \cdot (3^{-1})^2 \equiv 2 \cdot 67^2 \equiv 8978 \equiv \boxed{78} \pmod{100}.$$