1. Let $p$ be a prime number greater than 5. Prove that there exists a positive integer $n$ such that $p$ divides $20^n + 15^n - 12^n$.

Solution:

I claim that $n = p - 3$ works. Using the cool “Pythagorean triple” $\frac{1}{20^2} + \frac{1}{15^2} = \frac{1}{12^2}$, we have $20^{p-3} + 15^{p-3} - 12^{p-3} = \frac{20^{p-1} - 1}{20^2} + \frac{15^{p-1} - 1}{15^2} - \frac{12^{p-1} - 1}{12^2} = 9(20^{p-1} - 1) + 16(15^{p-1} - 1) - 25(12^{p-1} - 1)$.

Since we know that this fraction is an integer, to show that it is divisible by $p$ it suffices to check that the numerator is divisible by $p$ and the denominator is not. Since $p > 5$, $p$ is relatively prime to 12, 15, and 20, so by Fermat’s little theorem we see that $p$ divides the numerator. Also since $p > 5$ and $3600 = 2^4 \cdot 3^2 \cdot 5^2$, we see that $p$ does not divide the denominator. We conclude that $p$ divides $20^{p-3} + 15^{p-3} - 12^{p-3}$.

The intuition for this solution is quite simple: since $f(n) = 20^n + 15^n - 12^n$ equals 0 for $n = -2$ and $f(n) \text{ mod } p$ is periodic with some period dividing $p - 1$ for $n \geq 0$ (by Fermat’s little theorem), we ought to have $f(-2 + (p - 1)) \equiv 0 \pmod{p}$.

2. Let $a, b, c$ be real numbers such that $a + b + c = abc$. Prove that $\frac{1}{a^2 + 1} + \frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} \geq \frac{3}{4}$.

Solution:

As the condition and the inequality are invariant under the transformation $(a, b, c) \rightarrow (-a, -b, -c)$, we may assume that at most one of $a, b, c$ is negative, so WLOG let $a, b \geq 0$. Let $A, B \in [0, \frac{\pi}{2})$ be such that $\tan A = a$ and $\tan B = b$, and let $C = \pi - A - B$. Then, $c = \frac{a + b}{a b - 1} = -\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan (A + B) = \tan C$.

We have $\frac{1}{a^2 + 1} + \frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} = \frac{1}{\tan^2 A + 1} + \frac{1}{\tan^2 B + 1} + \frac{1}{\tan^2 C + 1} = \cos^2 A + \cos^2 B + \cos^2 C = \cos^2 A + \cos^2 B + (\sin A \sin B - \cos A \cos B)^2 = \cos^2 A + \cos^2 B + (1 - \cos^2 A)(1 - \cos^2 B) - 2 \sin A \sin B \cos A \cos B + \cos^2 A \cos^2 B = 1 - 2 \cos A \cos B (\sin A \sin B - \cos A \cos B) = 1 - 2 \cos A \cos B \cos C$.

We now show that $\cos A \cos B \cos C \leq \frac{1}{4}$. Since $A, B \in [0, \frac{\pi}{2})$, $\cos A \geq 0$ and $\cos B \geq 0$, so if $\cos C < 0$ then $\cos A \cos B \cos C < 0 \leq \frac{1}{4}$. Otherwise we may assume $\cos C \geq 0$, so by AM-GM we have $\cos A \cos B \cos C \leq \left(\frac{\cos A + \cos B + \cos C}{3}\right)^3$. Finally, since $A, B, C \in [0, \pi]$ and $\cos x$ is concave on this interval, we have by Jensen’s that $\cos A + \cos B + \cos C \leq \cos \left(\frac{A + B + C}{3}\right) = \cos \left(\frac{\pi}{3}\right) = \frac{1}{2}$. Putting this together gives $\cos A \cos B \cos C \leq \frac{1}{8}$, so $\frac{1}{a^2 + 1} + \frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} = 1 - 2 \cos A \cos B \cos C \geq \frac{3}{4}$.

3. Let $ABC$ be a triangle with incenter $I$, and let $D$ be the foot of the angle bisector from $A$ to $BC$. Let $\Gamma$ be the circumcircle of triangle $BIC$, and let $PQ$ be a chord of $\Gamma$ passing through $D$. Prove that $AD$ bisects $\angle PAQ$. 

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Solution:

If $PQ = BC$ the result is trivial, so we may assume otherwise.

Let $m\angle ABC = b$, $m\angle BCA = c$, and $m\angle CAB = a$. Also, let $T$ be the center of $\Gamma$ and let $\Omega$ be the circumcircle of triangle $ABC$. We first claim that $T$ lies on $\Omega$. As $m\angle ICB = \frac{c}{2}$, we have $m\angle ITB = c$. Similarly, $m\angle ITC = b$. Thus $m\angle BAC + m\angle BTC = a + (b + c) = \pi$, so quadrilateral $ACTB$ is cyclic. As $BT$ and $CT$ are chords of $\Omega$ with equal length, we must have $m\angle BAT = m\angle CAT$, so $T$ lies on line $\overrightarrow{AD}$.

We now wish to show that quadrilateral $AQTP$ is cyclic. Let $\Lambda$ be the circumcircle of triangle $APQ$. Since one of $P, Q$ lies inside $\Omega$ and the other lies outside $\Omega$, $\Lambda$ and $\Omega$ must intersect in exactly two points, and we let the point of intersection which is not $A$ be called $T'$. As $\Lambda$ and $\Omega$ have radical axis $\overrightarrow{AT'}$, $\Lambda$ and $\Gamma$ have radical axis $\overrightarrow{PQ}$, and $\Omega$ and $\Gamma$ have radical axis $\overrightarrow{BC}$, it follows by the radical axis theorem that these three line segments must be concurrent. As $\overrightarrow{BC}$ and $\overrightarrow{PQ}$ intersect at point $D$, we see that $\overrightarrow{AT'}$ must pass through $D$, so $T'$ lies both on $\overrightarrow{AD}$ and on $\Omega$. $\overrightarrow{AD}$ and $\Omega$ intersect only at $A$ and $T$, and as $T' \neq A$, it follows that $T' = T$. Thus $AQTP$ is cyclic.

As $|PT| = |QT|$, we see that $m\angle TPQ = m\angle TQP$. Since $AQTP$ is cyclic, we conclude that $m\angle DAP = m\angle TAP = m\angle TQP = m\angle TPQ = m\angle TAQ = m\angle DAQ$, and the result follows.

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