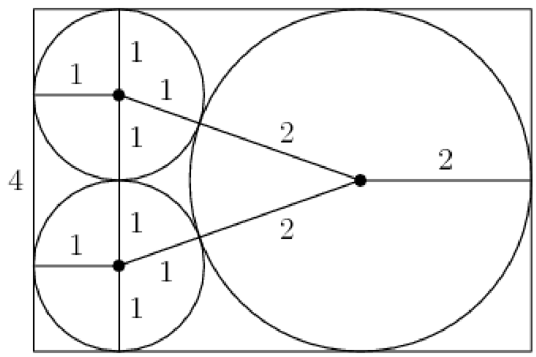
PUMaC 2012

Geometry A Solutions

Written by Ante Qu

1. [**3**] Three circles, with radii of $1, 1$, and 2, are externally tangent to each other. The minimum possible area of a quadrilateral that contains and is tangent to all three circles can be written as $a + b\sqrt{c}$ where c is not divisible by any perfect square larger than 1. Find $a + b + c$.

   Solution: In order to have the smallest quadrilateral, we want to make it tangent at as many points as possible. Consider the following quadrilateral.



   Note that 3 of the sides are fixed as 3 sides of a rectangle, so the area is proportional to the distance from the side tangential to the two small circles to the midpoint of the fourth side. This can be minimized by making the fourth side the fourth side of a rectangle, which is tangent to the large circle at its midpoint. To find the width of this rectangle, note that it is the sum of a radius of a small circle, a radius of the large circle, and the altitude of a isosceles triangle with side lengths of $2, 3, 3$. Using the Pythagorean Theorem, we find the altitude to be $2\sqrt{2}$. To calculate the whole area of the rectangle, we multiply its height, 4, by its width, $3 + 2\sqrt{2}$, to get $12 + 8\sqrt{2}$. So $a + b + c = 12 + 8 + 2 = \boxed{22}$.

2. [**3**] Two circles centered at O and P have radii of length 5 and 6 respectively. Circle O passes through point P. Let the intersection points of circles O and P be M and N. The area of triangle $\triangle MNP$ can be written in simplest form as a/b. Find $a + b$.

   Solution: Draw $\triangle MOP$ and $\triangle NOP$. They are each triangles with side lengths of $5, 5, 6$. Let V be the point of intersection of $\overline{MN}$ and $\overline{OP}$. Note that $\overline{MV}$ and $\overline{NV}$, the altitudes from $\overline{OP}$ to vertices M and N, are each equal to twice the area of each triangle divided by the length of side $\overline{OP}$. To find the area of $\triangle MOP$, simply take the altitude to $\overline{MP}$ and call the intersection at the base X. Since this triangle is isosceles, the altitude splits $\overline{MP}$ into two equal segments, so $\overline{XP}$ has a length of 3., so $\triangle OXP$ forms a 3-4-5 right triangle, and the altitude has a length of 4. As a result, $\overline{MV}$ has a length of $24/5$, and so $MV = 48/5$. Using the Pythagorean Theorem, we have that the length of $\overline{VP}$ is $18/5$, and the total area of $\triangle NOP$ is $432/25$, so $a + b = 432 + 25 = \boxed{457}$.

3. [**4**] Six ants are placed on the vertices of a regular hexagon with an area of 12. At each point in time, each ant looks at the next ant in the hexagon (in counterclockwise order), and measures

the distance, s, to the next ant. Each ant then proceeds towards the next ant at a speed of $\frac{s}{100}$ units per year. After T years, the ants' new positions are the vertices of a new hexagon with an area of 4. T is of the form $a \ln b$, where b is square-free. Find $a + b$.

Solution: At each moment in time, the velocity in the inward radial direction is $s \cos 60$, which is $s/2$. The distance from the ant to the center is s. Thus this is a continuous compound interest problem with the interest rate $r = -\frac{1}{200}$. The side length at any point in time, as a function of the original side length D, is $De^{-\frac{1}{200}t}$. Set this equal to $D/\sqrt{3}$, and we have $t = 100 \ln 3$. $a + b = 100 + 3 = \boxed{103}$

4. [**4**] A square is inscribed in an ellipse such that two sides of the square respectively pass through the two foci of the ellipse. The square has a side length of 4. The square of the length of the minor axis of the ellipse can be written in the form $a + b\sqrt{c}$ where a, b, and c are integers, and c is not divisible by the square of any prime. Find the sum $a + b + c$.

   Solution: Let a be the length of the major axis, b be the length of the minor axis, and c be the distance from the foci to the center of the ellipse.
   Since the sum of the distances from any point on the ellipse to the foci is $2a$, we can use a vertex of the square to calculate $2a$. We have

$$2a = 2 + 2\sqrt{5}$$

   so $a = 1 + \sqrt{5}$. Now using the relation $b^2 = a^2 - c^2$, we have

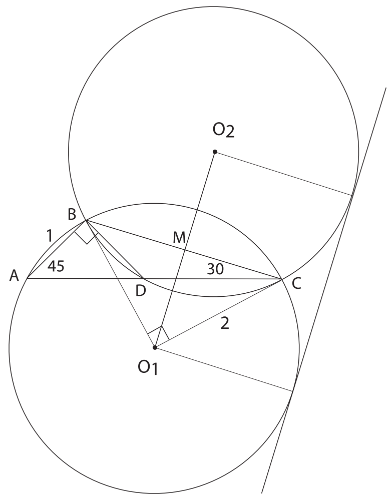
$$b^2 = 1 + 2\sqrt{5} + 5 - 4 = 2 + 2\sqrt{5}$$

$$2b = 2\sqrt{2 + 2\sqrt{5}} = \sqrt{8 + 8\sqrt{5}}$$

   So the square of this is $8 + 8\sqrt{5}$, so $a + b + c = 8 + 8 + 5 = \boxed{21}$
   Problem contributed by Elizabeth Yang

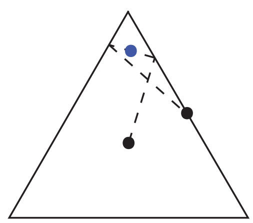
5. [**5**] Let $\triangle ABC$ be a triangle with $\angle BAC = 45°, \angle BCA = 30°$, and $AB = 1$. Point D lies on segment $\overline{AC}$ such that $AB = BD$. Find the square of the length of the common tangent between the circumcircles of triangles $\triangle BDC$ and $\triangle ABC$.

   Solution: We claim that the two circumcircles have same radius. Indeed, the circumcircle of ABC has radius $BC/2\sin(\angle BAC)$, and the circumcircle of BDC has radius $BC/2\sin(\angle BDC)$. Because $AB = BD$, we have that $m\angle BDC = 180° - m\angle BDA = 180° - m\angle BAC$, so $\angle BAC$ and $\angle BDC$ have the same sine, from which it follows that the two circumcircles have the same radius. Let O_1 and O_2 be the circumcenters of ABC and BDC. Then, the length of the common tangent is the same as the length of O_1O_2. The circumradius of ABC is $AB/(2\sin(30°)) = 1$. Note that $m\angle BO_1C = 2m\angle BAC = 90°$. If we let M be the intersection of BC and O_1O_2, then $O_1O_2 = 2O_1M = 2O_1C/\sqrt{2} = \sqrt{2}$, so the square of the length of the common tangent is $\boxed{2}$.

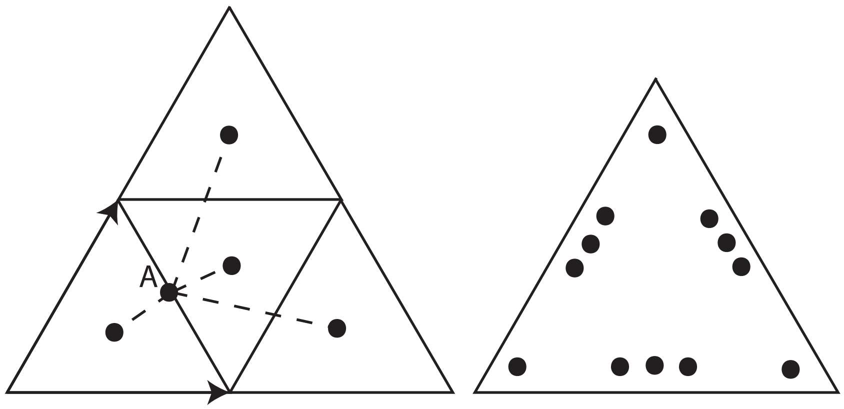


Problem contributed by Gene Katsevich

6. [**6**] Consider a pool table with the shape of an equilateral triangle. A ball of negligible size is initially placed at the center of the table. After it has been hit, it will keep moving in the direction it was hit towards and bounce off any edges with perfect symmetry. If it eventually reaches the midpoint of any edge, we mark the midpoint of the entire route that the ball has travelled through. Repeating this experiment, how many points can we mark at most?



Solution:
In the following diagram, the original triangle is the small triangle on the lower left corner of the left diagram, and note that two of its sides have been turned into vectors.



We firstly consider the midpoint A of one edge. We can keep reflecting the triangle along any edge, and the line connecting A and the center of any triangle in the plane represents a route between A and the center in the original triangle.

Define two vectors as shown in the diagram on the left: the horizontal one is 1, and the other one is v. A can be denoted as $\frac{1}{2} + \frac{1}{2}v$. For the triangles heading upwards (as in, triangles with the same orientation as the lower-left triangle), the center can be denoted as $a + \frac{1}{3} + \left(b + \frac{1}{3}\right) v$ with any integers a, b. Hence the midpoint of the route is $\frac{1}{2}a + \frac{5}{12} + \left(\frac{1}{2}b + \frac{5}{12}\right) v$. We can remove the integer parts, so there are four possible vectors starting at any vertex: $\frac{5}{12} + \frac{5}{12}v$, $\frac{11}{12} + \frac{5}{12}v$, $\frac{5}{12} + \frac{11}{12}v$, $\frac{11}{12} + \frac{11}{12}v$. But we need to consider the midpoints of all three different edges, and reflect them all back to the original triangle. Then we obtain the 12 points above.

For the triangles heading downwards (as in, with the same orientation as the middle triangle in the left diagram), the center can be denoted as $a + \frac{2}{3} + \left(b + \frac{2}{3}\right) v$. Eventually we obtain the same 12 points.

This conclusion also implies that $\boxed{12}$ points are enough to block all such routes.

Problem contributed by Xufan Zhang

7. [**7**] An octahedron (a solid with 8 triangular faces) has a volume of 1040. Two of the spatial diagonals intersect, and their plane of intersection contains four edges that form a cyclic quadrilateral. The third spatial diagonal is perpendicularly bisected by this plane and intersects the plane at the circumcenter of the cyclic quadrilateral. Given that the side lengths of the cyclic quadrilateral are $7, 15, 24, 20$, in counterclockwise order, the sum of the side lengths of the entire octahedron can be written in simplest form as a/b. Find $a + b$.

Solution: Note that $7^2 + 24^2 = 15^2 + 20^2$, so the cyclic quadrilateral has perpendicular diagonals. For a quadrilateral with perpendicular diagonals, the area is $K = pq/2$ where p and q are the lengths of the diagonals. Thus we can compute the area using Ptolemy's theorem: $K = pq/2 = (ac + bd)/2$ where a, b, c, d are consecutive lengths of the sides. Plugging in, we have the area $K = 234$. The volume of the octahedron is the sum of the volumes of two pyramids, each with the quadrilateral as the base and half of the third diagonal as the height. So volume of the octahedron is $V = KL/3$ where L is the length of the third diagonal. Solving for L, the length of the third spatial diagonal is $3 * 1040/234 = 40/3$.

Label the quadrilateral as $ABCD$ (such that $a = AB$, $b = BC$, $c = CD$, and $d = DA$), and let O be the center of the circumcircle of this quadrilateral. To get the circumradius, we consider the fact that the arcs of opposite sides add up to $180°$ because the diagonals are perpendicular. This implies that $\angle AOB$ is supplementary to $\angle COD$, so $\cos\angle AOB = -\cos\angle COD$. Using the law of cosines on $\triangle AOB$ and $\triangle COD$ and adding, we have $4R^2 = a^2 + c^2$. So the circumradius is $R = \sqrt{(a^2+c^2)/4} = 25/2$. Using the pythagorean theorem, we can find each missing side length by taking R to be one leg, half of L to be the other leg, and the missing side length to be the hypoteneuse. Thus each missing side length has a length of $85/6$, and the whole octahedron has 8 of these plus the 4 sides of the quadrilateral, summing to $340/3 + 66 = 538/3$. Thus $a + b = \boxed{541}$.

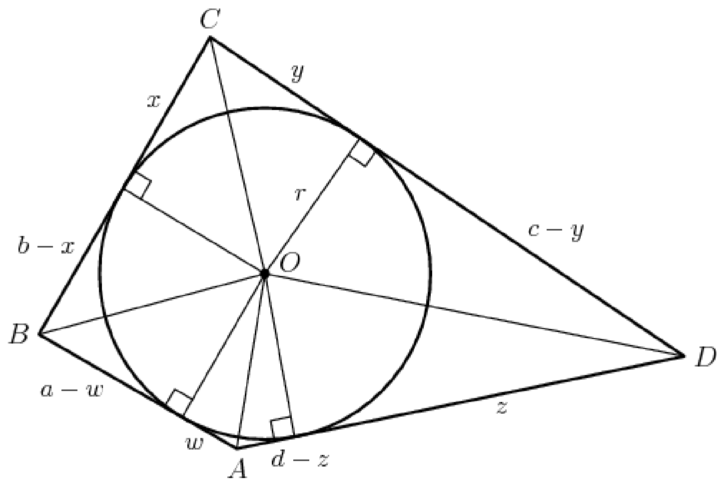
Interesting alternate way to calculate the circumradius and the area of the quadrilateral:
We can also note that the circumradius and the area of a quadrilateral does not change if we swap two of its side lengths. Therefore we can consider the cyclic quadrilateral with side lengths $7, 24, 15, 20$, and note that one of its diagonals is the hypoteneuse of two pythagorean triples (This can be shown via the law of cosines on the two triangles formed by this diagonal, and noting the cosines of opposite vertices (which are supplementary) have the same value but opposite sign.), so it has length 25, and the two vertices opposite it are right angles. Thus the diagonal is a diameter, and the circumradius is half of that, or $25/2$. The area can also be computed as the sum of the areas of the two right triangles, which is $84 + 150 = 234$.

8. [**8**] Cyclic quadrilateral $ABCD$ has side lengths $AB = 2$, $BC = 3$, $CD = 5$, $AD = 4$. Find $\sin A \sin B(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} + \cot\frac{D}{2})^2$. Your answer can be written in simplest form as a/b. Find $a + b$.

Solution:
(This is somewhat similar to 2006 Geometry #8)
Label side AB as a, BC as b, CD as c, AD as d.

Label diagonal BD as f, and the circumradius as R. Then the area of triangle ABC is $\frac{1}{2}ab\sin B$, and the area of triangle BCD is $\frac{1}{2}cd\sin D$. Since B and D are supplementary, $\sin B = \sin D$. So the area of the quadrilateral, K, is $\frac{1}{2}(ab+cd)\sin B$. Similarly, if we split it via the other diagonal, we get $K = \frac{1}{2}(bc+ad)\sin A$. So $\sin A \sin B = \dfrac{4K^2}{(ab+cd)(bc+ad)}$

Note that since $a + c = b + d$, the quadrilateral is tangential (in other words, it contains an incircle). Draw the incircle with inradius r and center O:



Since $\overline{AO}$, $\overline{BO}$, $\overline{CO}$, and $\overline{DO}$ are angle bisectors, we have:

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}+\cot\frac{D}{2}$$
$$=\frac{1}{2}\left(\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)+\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)+\left(\cot\frac{C}{2}+\cot\frac{D}{2}\right)+\left(\cot\frac{D}{2}+\cot\frac{A}{2}\right)\right)$$
$$=\frac{1}{2}\left(\left(\frac{w+(a-w)}{r}\right)+\left(\frac{x+(b-x)}{r}\right)+\left(\frac{y+(c-y)}{r}\right)+\left(\frac{z+(d-z)}{r}\right)\right)$$
$$=\frac{a+b+c+d}{2r}$$

In addition, $K=rs$, where s is the semi perimeter. So $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}+\cot\frac{D}{2}=\frac{(a+b+c+d)^2}{4K}$.

Multiplying what we have, the final result is $\frac{4K^2}{(ab+cd)(bc+ad)}\frac{(a+b+c+d)^4}{16K^2}=\frac{(a+b+c+d)^4}{4(ab+cd)(bc+ad)}$. Plugging in the original numbers, we have $4802/299$. So $a+b=\boxed{5101}$.

Alternate solution (by Elizabeth Yang):
We can start by simplifying $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}+\cot\frac{D}{2}$ using the half angle formula

$$\cot\frac{x}{2}=\frac{1+\cos x}{\sin x}$$

This gives

$$\frac{1+\cos A}{\sin A}+\frac{1+\cos B}{\sin B}+\frac{1+\cos C}{\sin C}+\frac{1+\cos D}{\sin D}$$

Since $ABCD$ is cyclic, $\cos A=-\cos C$ and $\cos B=-\cos C$, so we can cancel terms out.

$$\frac{2}{\sin A}+\frac{2}{\sin B}$$

Squaring and then multiplying by $(\sin A)(\sin B)$, we have

$$8+4\left(\frac{\sin A}{\sin B}+\frac{\sin B}{\sin A}\right)$$

Using Law of Sines on triangles ABD and ABC and the fact that $\angle BDA \cong \angle BCA$, we get the ratio

$$\frac{AC}{\sin A}=\frac{BD}{\sin B}$$

$$\frac{\sin A}{\sin B}=\frac{AC}{BD}$$

Going back to the expression from the problem, we now have

$$8+4\left(\frac{AC}{BD}+\frac{BD}{AC}\right)$$

We can sub in $\frac{22}{AC}$ for BD (from Ptolemy's), giving us

$$8+4\left(\frac{(AC)^2}{22}+\frac{22}{(AC)^2}\right)$$

Using Law of Cosines on triangles BCA and ACD to solve for AC and plugging in, we get $\frac{4802}{299}$. So $a+b=\boxed{5101}$.