PUMaC 2012

Geometry A

1. [**3**] Three circles, with radii of $1, 1$, and 2, are externally tangent to each other. The minimum possible area of a quadrilateral that contains and is tangent to all three circles can be written as $a + b\sqrt{c}$ where c is not divisible by any perfect square larger than 1. Find $a + b + c$.

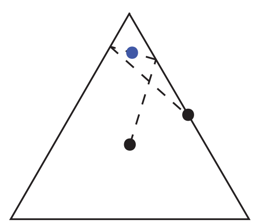
2. [**3**] Two circles centered at O and P have radii of length 5 and 6 respectively. Circle O passes through point P. Let the intersection points of circles O and P be M and N. The area of triangle $\triangle MNP$ can be written in simplest form as a/b. Find $a + b$.

3. [**4**] Six ants are placed on the vertices of a regular hexagon with an area of 12. At each point in time, each ant looks at the next ant in the hexagon (in counterclockwise order), and measures the distance, s, to the next ant. Each ant then proceeds towards the next ant at a speed of $\frac{s}{100}$ units per year. After T years, the ants' new positions are the vertices of a new hexagon with an area of 4. T is of the form $a \ln b$, where b is square-free. Find $a + b$.

4. [**4**] A square is inscribed in an ellipse such that two sides of the square respectively pass through the two foci of the ellipse. The square has a side length of 4. The square of the length of the minor axis of the ellipse can be written in the form $a + b\sqrt{c}$ where a, b, and c are integers, and c is not divisible by the square of any prime. Find the sum $a + b + c$.

5. [**5**] Let $\triangle ABC$ be a triangle with $\angle BAC = 45°, \angle BCA = 30°$, and $AB = 1$. Point D lies on segment $\overline{AC}$ such that $AB = BD$. Find the square of the length of the common external tangent to the circumcircles of triangles $\triangle BDC$ and $\triangle ABC$.

6. [**6**] Consider a pool table with the shape of an equilateral triangle. A ball of negligible size is initially placed at the center of the table. After it has been hit, it will keep moving in the direction it was hit towards and bounce off any edges with perfect symmetry. If it eventually reaches the midpoint of any edge, we mark the midpoint of the entire route that the ball has travelled through. Repeating this experiment, how many points can we mark at most?



7. [**7**] An octahedron (a solid with 8 triangular faces) has a volume of 1040. Two of the spatial diagonals intersect, and their plane of intersection contains four edges that form a cyclic quadrilateral. The third spatial diagonal is perpendicularly bisected by this plane and intersects the plane at the circumcenter of the cyclic quadrilateral. Given that the side lengths of the cyclic quadrilateral are $7, 15, 24, 20$, in counterclockwise order, the sum of the edge lengths of the entire octahedron can be written in simplest form as a/b. Find $a + b$.

8. [**8**] Cyclic quadrilateral $ABCD$ has side lengths $AB = 2$, $BC = 3$, $CD = 5$, $AD = 4$. Find $\sin A \sin B(\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} + \cot \frac{D}{2})^2$. Your answer can be written in simplest form as a/b. Find $a + b$.