PUMaC 2012

Combinatorics A

1. [**3**] If the probability that the sum of three distinct integers between 16 and 30 (inclusive) is even can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

2. [**3**] How many ways are there to arrange the 6 permutations of the tuple $(1,2,3)$ in a sequence, such that each pair of adjacent permutations contains at least one entry in common? For example, a valid such sequence is given by
$$(3,2,\mathbf{1})-(\mathbf{2},3,1)-(2,1,\mathbf{3})-(\mathbf{1},2,3)-(1,3,\mathbf{2})-(3,1,2).$$

3. [**4**] Jim has two fair 6-sided dice, one whose faces are labelled from 1 to 6, and the second whose faces are labelled from 3 to 8. Twice, he randomly picks one of the dice (each die equally likely) and rolls it. Given the sum of the resulting two rolls is 9, if $\frac{m}{n}$ is the probability he rolled the same die twice where m,n are relatively prime positive integers, then what is $m+n$?

4. [**4**] How many (possibly empty) sets of lattice points $\{P_1,P_2,\ldots,P_M\}$, where each point $P_i=(x_i,y_i)$ for $x_i,y_i\in\{0,1,2,3,4,5,6\}$, satisfy that the slope of the line P_iP_j is positive for each $1\le i<j\le M$? An infinite slope, e.g. P_i is vertically above P_j, does not count as positive.

5. [**5**] 5 people stand in a line facing one direction. In every round, the person at the front moves randomly to any position in the line, including the front or the end. Suppose that $\frac{m}{n}$ is the expected number of rounds needed for the last person of the initial line to appear at the front of the line, where m and n are relatively prime positive integers. What is $m+n$?

6. [**6**] Two white pentagonal pyramids, with side lengths all the same, are glued to each other at their regular pentagon bases. Some of the resulting 10 faces are colored black. How many rotationally distinguishable colorings may result?

7. [**7**] A PUMaC grader is grading the submissions of forty students $s_1,s_2,\ldots,s_{40}$ for the individual finals round, which has three problems. After grading a problem of student s_i, the grader either:

   - grades another problem of the same student, or
   - grades the same problem of the student s_{i-1} or s_{i+1} (if $i>1$ and $i<40$, respectively).

   He grades each problem exactly once, starting with the first problem of s_1 and ending with the third problem of s_{40}. Let N be the number of different orders the grader may grade the students' problems in this way. Find the remainder when N is divided by 100.

8. [**8**] Proctors Andy and Kristin have a PUMaC team of eight students labelled $s_1,s_2,\ldots,s_8$ (the PUMaC staff being awful with names). The following occurs:
   1. Andy tells the students to arrange themselves in a line in arbitrary order.
   2. Kristin tells each student s_i to move to the current spot of student s_j, where $j\equiv 3i+1 \pmod 8$.
   3. Andy tells each student s_i to move to the current spot of the student who was in the ith position of the line after step 1.

   How many possible orders can the students be in now?