PUMaC 2012

Algebra A

1. [**3**] Compute the smallest positive integer a for which $\sqrt{a+\sqrt{a+\ldots}} - \frac{1}{a+\frac{1}{a+..}} > 7$.

2. [**3**] If x, y, and z are real numbers with $\frac{x-y}{z} + \frac{y-z}{x} + \frac{z-x}{y} = 36$, find
$$2012 + \frac{x-y}{z} \cdot \frac{y-z}{x} \cdot \frac{z-x}{y}$$

3. [**4**] Compute
$$\sum_{n=1}^{\infty} \frac{n+1}{n^2(n+2)^2}$$
Your answer in simplest form can be written as a/b, where a, b are relatively-prime positive integers. Find $a+b$.

4. [**4**] Let f be a polynomial of degree 3 with integer coefficients such that $f(0) = 3$ and $f(1) = 11$. If f has exactly 2 integer roots, how many such polynomials f exist?

5. [**5**] What is the smallest natural number n greater than 2012 such that the polynomial $f(x) = (x^6 + x^4)^n - x^{4n} - x^6$ is divisible by $g(x) = x^4 + x^2 + 1$?

6. [**6**] Let a_n be a sequence such that $a_0 = 0$ and:
$$a_{3n+1} = a_{3n} + 1 = a_n + 1$$
$$a_{3n+2} = a_{3n} + 2 = a_n + 2$$
for all natural numbers n. How many n less than 2012 have the property that $a_n = 7$?

7. [**7**] Let a_n be a sequence such that $a_1 = 1$ and $a_{n+1} = \lfloor a_n + \sqrt{a_n} + \frac{1}{2} \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. What are the last four digits of a_{2012}?

8. [**8**] If n is an integer such that $n \geq 2^k$ and $n < 2^{k+1}$, where $k = 1000$, compute the following:
$$n - \left(\left\lfloor \frac{n-2^0}{2^1} \right\rfloor + \left\lfloor \frac{n-2^1}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{n-2^{k-1}}{2^k} \right\rfloor \right).$$