PUMaC 2011

Team Round

Time Limit: 25 minutes *Maximum Possible Score: 81*

The following is a mathematical Sudoku puzzle which is also a crossword. Your job is to fill in as many blanks as you possibly can, **including all shaded squares**. You **do not** earn extra points for showing your work; the only points you get are for correctly filled-in squares. You get **one point for each correctly filled-in square**. You should read through the following rules carefully before starting.

- Your **time limit** for this round is 25 minutes, in addition to the five minutes you get for reading the rules. So make use of your time wisely. The round is based more on speed than on perfect reasoning, so use your intuition well, and **be fast**.
- This is a **Sudoku puzzle**; all the squares should be filled in with the digits 1 through 9 so that every row and column contains each digit exactly once. In addition, each of the nine 3×3 boxes that compose the grid also contains each digit exactly once. Furthermore, this is a **super-Sudoku puzzle**; in addition to satisfying all these conditions, the four 3×3 boxes with red outlines also contain each of $1, \ldots, 9$ exactly once. This last property is important to keep in mind – it may help you solve the puzzle faster.
- Just to restate the idea, you can use the digits 1 through 9, but not 0. You may not use any other symbol, such as π or e or ε. Each square gets exactly one digit.
- The grid is also a **crossword puzzle**; the usual rules apply. The shaded grey squares are the "black" squares of an ordinary crossword puzzle. The white squares as well as the shaded yellow ones count as the "white" crossword squares. **All squares, white or shaded, count as ordinary Sudoku squares.**
- If you obtain the unique solution to the crossword puzzle, then this solution extends to a **unique** solution to the Sudoku puzzle.
- You **may** use a graphing calculator to help you solve the clues.

The following hints and tips may prove useful while solving the puzzle.

- **Use the super-Sudoku structure** described in the first rule; use all the symmetries you have. Remember that we are not looking for proofs or methods, only for correctly filled-in squares.
- If you find yourself stuck on a specific clue, it is nothing to worry about. You can obtain the solution to that clue later on by solving other clues and figuring out certain digits of your desired solution. Just move on to the rest of the puzzle.
- As you progress through the puzzle, keep filling in all squares you have found on your solution sheet, **including the shaded ones**. Remember that for scoring, the shaded grey squares count the same as the white ones.

Good luck!

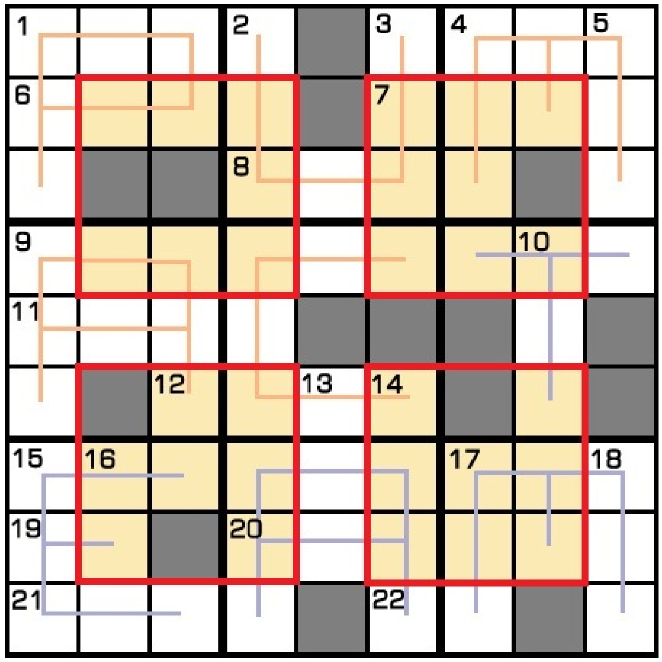


Figure 1: Good luck. Make sure you turn in the solution sheet, not this page!

1 Across

- **1 Across.** The following is a normal addition where each letter represents a (distinct) digit: $GOT + TO + GO + TO = TOP$. This certainly does not have a unique solution. However, you discover suddenly that $G = 2$ and $P \notin \{4, 7\}$. Then what is the numeric value of the expression $GOT \times TO$?

- **3 Across.** A **strobogrammatic** number which reads the same upside down, e.g. 619. On the other hand, a **triangular number** is a number of the form $n(n+1)/2$ for some $n \in \mathbb{N}$, e.g. 15 (therefore, the i^{th} triangular number T_i is the sum of 1 through i). Let a be the **third** strobogrammatic prime number. Let b be the smaller number of the **smallest** pair of triangular numbers whose sum and difference are also triangular numbers. What is the value of ab?

- **6 Across.** A positive integer m is said to be **palindromic** in base ℓ if, when written in base ℓ, its digits are the same front-to-back and back-to-front. For $j, k \in \mathbb{N}$, let $\mu(j, k)$ be the smallest base-10 integer that is palindromic in base j as well as in base k, and let $\nu(j, k) := (j + k) \cdot \mu(j, k)$. Find the value of $\nu(5, 9)$.

- **7 Across.** Suppose you have the unique solution to this Sudoku puzzle. In that solution, let $\mathfrak{X}$ denote the sum of all digits in the shaded grey squares. Similarly, let $\mathfrak{Y}$ denote the sum of all numbers in the shaded yellow squares on the **upper left** block (i.e. the 3×3 box outlined red towards the top left). Concatenate $\mathfrak{X}$ with $\mathfrak{Y}$ in that order, and write that down.

- **8 Across.** For any $n \in \mathbb{N}$ such that $1 < n < 10$, define the sequence $X_{n,1}, X_{n,2}, \ldots$ by $X_{n,1} = n$, and for $r \geq 2$, $X_{n,r}$ is smallest number $k \in \mathbb{N}$ larger than $X_{n,r-1}$ such that k and the sum of digits of k are *both* powers of n. For instance, $X_{3,1} = 3$, $X_{3,2} = 9$, $X_{3,3} = 27$, and so on. Concatenate $X_{2,2}$ with $X_{2,4}$, and write down the answer.

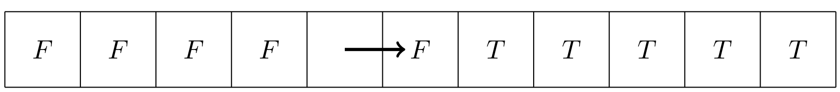
- **9 Across.** Find positive integers x, y, z satisfying the following properties: y is obtained by subtracting 93 from x, and z is obtained by subtracting 183 from y; furthermore, x, y and z in their base-10 representations contain precisely all the digits from 1 through 9 once (i.e. concatenating x, y and z yields a valid 9-digit Sudoku answer). Obviously, write down the concatenation of x, y and z in that order.

- **11 Across.** Find the **largest** pair of two-digit consecutive prime numbers a and b (with $a < b$) such that the sum of the digits of a plus the sum of the digits of b is also a prime number. Write the concatenation of a and b.

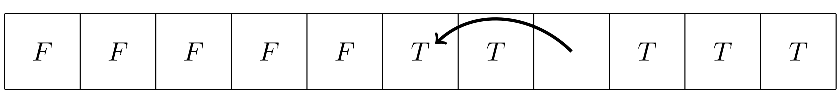
- **12 Across.** Suppose you have a strip of $2n + 1$ squares, with n frogs on the n squares on the left, and n toads on the n squares on the right. A move consists either of a toad or a frog sliding to an adjacent square if it is vacant, or of a toad or a frog jumping one square over another one and landing on the next square if it is vacant. For instance, the starting position

F	F	F	F	F		T	T	T	T	T

has the position



or the position



as results of valid first moves. What is the minimum number of moves needed to swap the toads with the frogs if $n = 5$? How about $n = 6$? Concatenate your answers.

- **15 Across.** Let w be the largest number such that w, $2w$ and $3w$ together contain every digit from 1 through 9 exactly once. Let x be the smallest integer with the property that its first 5 multiples contain the digit 9. A **Leyland number** is an integer of the form $m^n + n^m$ for integers $m, n > 1$. Let y be the fourth Leyland number. A **Pillai prime** is a prime number p for which there is an integer $n > 0$ such that $n! \equiv -1 \pmod{p}$, but $p \not\equiv 1 \pmod{n}$. Let z be the fourth Pillai prime. Concatenate w, x, y and z in that order to obtain a permutation of $1, \cdots, 9$. Write down this permutation.

- **19 Across.** A **hoax number** $k \in \mathbb{N}$ is one for which the sum of its digits (in base 10) equals the sum of the digits of its distinct prime factors (in base 10). For instance, the distinct prime factors of 22 are 2 and 11, and we have $2+2 = 2+(1+1)$. In fact, 22 is the first hoax number. What is the second?

- **20 Across.** Let a, b and c be distinct 2-digit numbers satisfying the following properties:

  - a is the **largest** integer expressible as $a = x^y = y^x$, for distinct integers x and y.
  - b is the **smallest** integer which has three partitions into three parts, which all give the same product (which turns out to be 1200) when multiplied.
  - c is the **largest** number that is the sum of the digits of its cube.

  Concatenate a, b and c, and write down the resulting 6-digit prime number.

- **21 Across.** Suppose $N = \underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$ is a 4-digit number with digits a, b, c and d, such that $N = a \cdot b \cdot c \cdot d^7$. Find N.

- **22 Across.** What is the smallest number expressible as the sum of 2, 3, 4, or 5 **distinct** primes?

2 Down

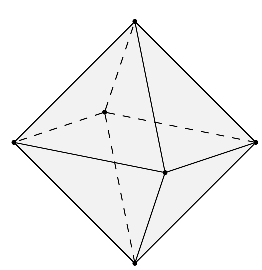
- **1 Down.** For some $a, b, c \in \mathbb{N}$, let the polynomial

  $$p(x) = x^5 - 252x^4 + ax^3 - bx^2 + cx - 62604360$$

  have five distinct roots that are positive integers. Four of these are 2-digit numbers, while the last one is single-digit. Concatenate all five roots in **decreasing** order, and write down the result.

- **2 Down.** Gene, Ashwath and Cosmin together have 2511 math books. Gene now buys as many math books as he already has, and Cosmin sells off half his math books. This leaves them with 2919 books in total. After this, Ashwath suddenly sells off all his books to buy a private jet, leaving Gene and Cosmin with a total of 2184 books. How many books did Gene, Ashwath and Cosmin have to begin with? Concatenate the three answers (in the order Gene, Ashwath, Cosmin) and write down the result.

- **3 Down.** A **regular octahedron** is a convex polyhedron composed of eight congruent faces, each of which is an equilateral triangle; four of them meet at each vertex. For instance, the following diagram depicts a regular octahedron:



Let $\mathscr{T}$ be a regular octahedron of edge length 28. What is the total surface area of $\mathscr{T}$, rounded to the nearest integer?

- **4 Down.** Evaluate the value of the expression

$$\sum_{k=T_{24}+1}^{T_{25}} k,$$

where T_i denotes the i^{th} triangular number (the sum of the integers from 1 through i).

- **5 Down.** Suppose r and s are consecutive multiples of 9 satisfying the following properties:
  - r is the smallest positive integer that can be written as the sum of 3 positive squares in 3 **different** ways.
  - s is the smallest 2-digit number that is a Woodall number as well as a base-10 Harshad number. A **Woodall number** is any number of the form $n \cdot 2^n - 1$ for some $n \in \mathbb{N}$. A **base-**10 **Harshad number** is divisible by the sum of its digits in base 10.

Concatenate r and s and write down the result.

- **10 Down.** For any $k \in \mathbb{N}$, let $\varphi_p(k)$ denote the sum of the distinct prime factors of k. Suppose N is the largest integer less than 50000 satisfying $\varphi_p(N) = \varphi_p(N+1)$, where the common value turns out to be a meager 55. What is N?

- **13 Down.** The n^{th} s**-gonal number** $P(s,n)$ is defined as

$$P(s,n) = (s-3)T_{n-1} + T_n,$$

where T_i is the i^{th} triangular number (recall that the i^{th} triangular number is the sum of the numbers 1 through i). Find the least N such that N is both a 34-gonal number, *and* a 163-gonal number.

- **14 Down.** A **biprime** is a positive integer that is the product of precisely two (not necessarily distinct) primes. A **cluster** of biprimes is an ordered triple $(m, m+1, m+2)$ of consecutive integers that are biprimes. There are precisely three clusters of biprimes below 100. Denote these by, say,

$$\{(p,p+1,p+2),(q,q+1,q+2),(r,r+1,r+2)\},$$

and add the condition that $p+2 < q < r-2$ to fix the three clusters. Interestingly, $p+1$ and q are both multiples of 17. Concatenate q with $p+1$ in that order, and write down the result.

- **16 Down.** Find the least positive integer m (written in base 10 as $m = \underline{a}\,\underline{b}\,\underline{c}$, with digits a, b, c), such that $m = (b+c)^a$.

- **17 Down.** Let $\mathfrak{X}$ be a set containing 32 elements, and let $\mathfrak{Y} \subseteq \mathfrak{X}$ be a subset containing 29 elements. How many 2-element subsets of $\mathfrak{X}$ are there which have nonempty intersection with $\mathfrak{Y}$?

- **18 Down.** Find a positive integer $\mathfrak{K} < 196$, which is a strange twin of the number 196, in the sense that $\mathfrak{K}^2$ shares the same digits as 196^2, and $\mathfrak{K}^3$ shares the same digits as 196^3.