PUMaC 2011

Geometry B Solutions

1. Because $\angle A = 70°$, we know that $\angle ABH = 20°$, so $\angle HBC = 40°$. Constructing DC, we have that triangle BDC is isosceles, so $\angle BDC = \angle BCD = 70°$. Noticing that $\angle BAC = \angle BDC = 70°$, we have that quadrilateral $ABCD$ is cyclic. It follows that $\angle BDA = \angle BCA = \boxed{50°}$.

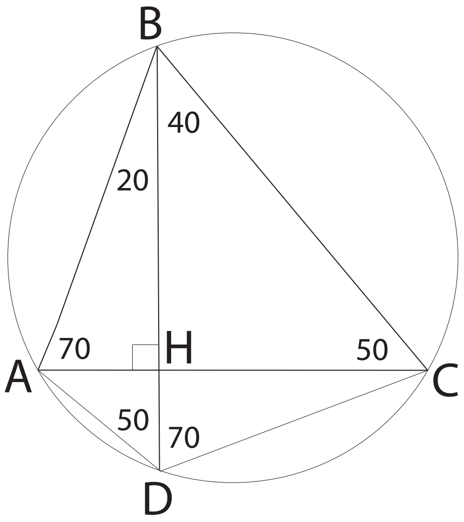

Figure 1: Problem 1 diagram.

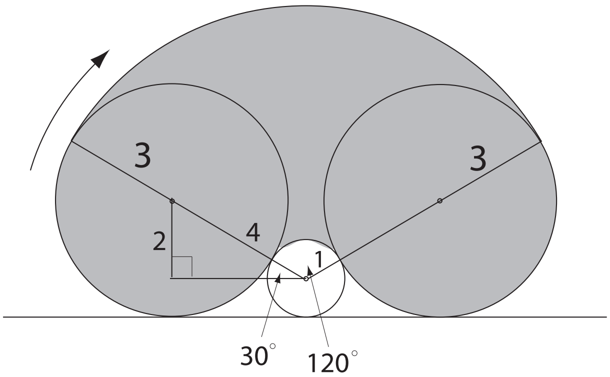

Figure 2: Problem 2 diagram.

2. Note that the solid formed is a generalized cylinder. It is clear from the diagram that the area of the base of this cylinder (i.e., a vertical cross-section of the log) is composed of two semicircles of radius 3 and a part of an annulus. In the right triangle in the diagram, the hypotenuse is 4 and the vertical leg is 2. Thus, it is a 30-60-90 triangle, so the central angle in

the annulus is 120°. Since the annular region has inner radius 1 and outer radius 7, the total area is $2(\frac{1}{2}\pi 3^2) + \frac{1}{3}\pi(7^2 - 1^2) = 25\pi$. Hence the volume of the cylinder is $10 \cdot 25\pi = 250\pi$, so the answer is $\boxed{250}$.

3. **First Solution**: Since $BC = (1/2)AD$, we have that $BC = MD$, and it follows that $\triangle BCP \cong \triangle DMP$. Thus, $CP = PM$. Select R on CD such that MR is parallel to AQ. Then, $CP = PM \implies CQ = QR$ and $AM = MD \implies QR = RD$. Thus, $CQ/QD = 1/2$, so the answer is $1 + 2 = \boxed{3}$.
Second Solution: As in the first solution, note that $\triangle BCP \cong \triangle DMP$. From this congruence it follows that $BP = PD$ and $CP = PM$. Extend AP to meet line BC at point A'. Because $CP = PM$, we have $\triangle APM \cong \triangle A'PC$. Thus, $BC = AM = CA'$. It follows that point Q is the centroid of triangle $BA'D$, so $CQ/QD = 1/2$. Thus, our answer is $1 + 2 = \boxed{3}$.

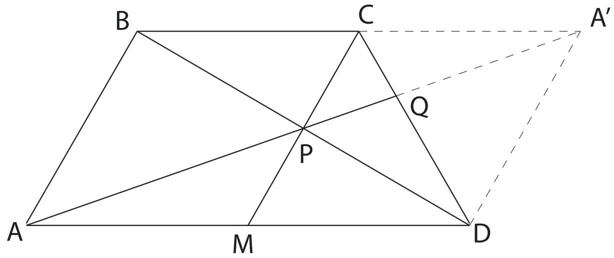

Figure 3: Problem 3 diagram.

4. It is easy to see, by the Pythagorean theorem, that $\mathcal{L}(c)$ for any c consists of a line perpendicular to $\overline{AB}$. Thus, in order for the intersection of $\mathcal{L}(c)$ and ω to consist of a single point, $\mathcal{L}(c)$ must be tangent to ω. In this case, define X to be the point on $\mathcal{L}(c)$ collinear with A, B. If B is between X and A then

$$\begin{aligned} c = (XA)^2 - (XB)^2 &= (XA - XB)(XA + XB) \\ &= (AB)(2r) \\ &= 5 \cdot 2 \cdot 6 \\ &= 60. \end{aligned}$$

Note that r above denotes the radius of ω. Otherwise, if A is between X and B then

$$(XA)^2 - (XB)^2 = -(AB)(2r) = -60.$$

Thus the possible values of c are ± 60, so our answer is $\boxed{c = 60}$.

5. **Note:** The problem was flawed as stated on the exam. Many thanks to Will Zhang of PEA Green for pointing out that there is a configuration of the three circles of given radii that can give rise to arbitrarily large radii for the fourth circle. If the problem were reworded to specify that the three circles with given radii were externally tangent to one another, the following would have been the solution:

The largest possible radius of the fourth circle is achieved when it is internally tangent to the first three. Let O_1 and O_2 be the centers of the circles of radius 5 and let O_3 be the center of the circle of radius 8. Let O be the center of the largest circle. Note that O must be on the altitude O_3H of the triangle $O_1O_2O_3$. Let r be the radius of the largest circle, and let $\theta = \angle OO_3O_2$. Note that O_2HO_3 is a 5-12-13 right triangle, so $HO_3 = 12$. From this right triangle, we find $\cos\theta = 12/13$. Then, from the theorem of cosines in triangle OO_3O_2 we find that

$$(r-8)^2 + 13^2 - 2(r-8)13 \cdot \frac{12}{13} = (r-5)^2.$$

Simplifying the above equation yields

$$\begin{aligned} &r^2 - 16r + 8^2 + 13^2 - 24(r-8) = r^2 - 10r + 5^2 \\ \Rightarrow &8^2 + 13^2 - 5^2 + 8 \cdot 24 = 30r \\ \Rightarrow &30r = 8^2 + 12^2 + 8 \cdot 24 = 16(4+9+12) = 16 \cdot 25 \Rightarrow r = \frac{40}{3}. \end{aligned}$$

Thus the answer is $40 + 3 = \boxed{43}$.

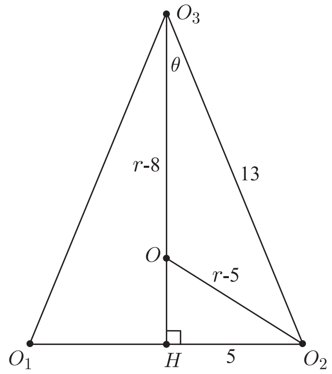

Figure 4: Problem 5 diagram.

6. Pick P on DM and R on CM so the AP is perpendicular to DM and BR is perpendicular to CM. Because of the way the paper is being folded, the projection of A onto the plane of the paper is always along line AP, and the projection of B along line BR. Thus, the two lines will intersect in exactly the point H. Since $\triangle HMB \sim \triangle MBC$, we have $HM/MB = MB/BC$, so $HM = (MB/BC) \cdot MB = (60/80) \cdot 60 = (3/4) \cdot 60 = \boxed{45}$.

7. Without loss of generality, suppose A lies to the left of B. Let D' be the point such that $DAD'B$ is a parallelogram. No matter what the positions of A and B are, we have that $BD = 15/\sin(60°) = 10\sqrt{3}$, $AC = 15/\sin(30°) = 30$, and $\angle CAD' = \angle CAB + \angle BAD' = \angle CAB + \angle DBA = 90°$. Thus, CD' is always $20\sqrt{3}$ as A and B vary. Note that $AD + BC =$

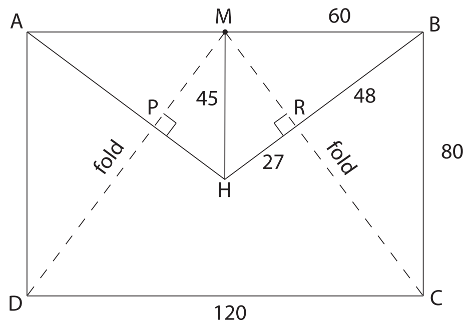

Figure 5: Problem 6 diagram.

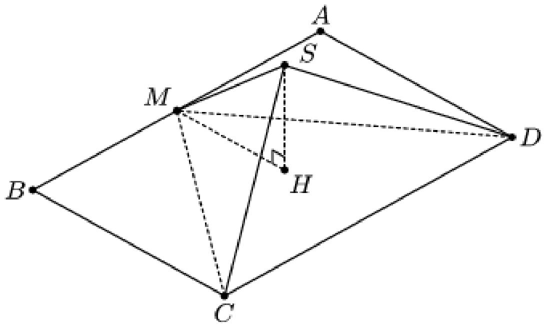

Figure 6: Problem 6 Diagram

$BD' + BC$. By the triangle inequality, this length is no less that $CD' = 20\sqrt{3}$, and equality can be achieved by fixing A and moving B to the intersection of CD' with ℓ_1. Thus, $20\sqrt{3}$ is the minimum length, so the answer is $20 + 3 = \boxed{23}$.

8. We claim that the length of arc MN is constant as P varies. We can see this by noting that $\overset{\frown}{MLB} - \overset{\frown}{AN} = \frac{1}{2}\angle APB$, which is constant, and that $\overset{\frown}{MLB} + \overset{\frown}{MA}$ is constant. Subtracting these two constant quantities, we get that $\overset{\frown}{MN} = \overset{\frown}{MA} + \overset{\frown}{AN}$ is constant. Since OS is the distance from O to the midpoint of a chord of constant length, OS is constant as well. Thus, the locus of all points S is a part of a circle centered at O. It follows that the minimum distance from this locus to point A is the difference between the radii of ω_1 and of the locus of S. Now, to find the radius of the locus of S, consider the location of S when P is at the midpoint C of major arc AB. Since ω_2 passes through O, we have that CA and CB are tangent to ω_1. Thus, the segment MN becomes AB, and S coincides with T, the midpoint of AB. By the similarity of triangles TAO and ACO, we have that $OT/OA = OA/OC$, so $OT = OA^2/OC = 6^2/10 = 18/5$. Thus, the radius of the locus of S is $18/5$, and the difference between the two radii is $6 - 18/5 = 12/5$, so the answer is $12 + 5 = \boxed{17}$.

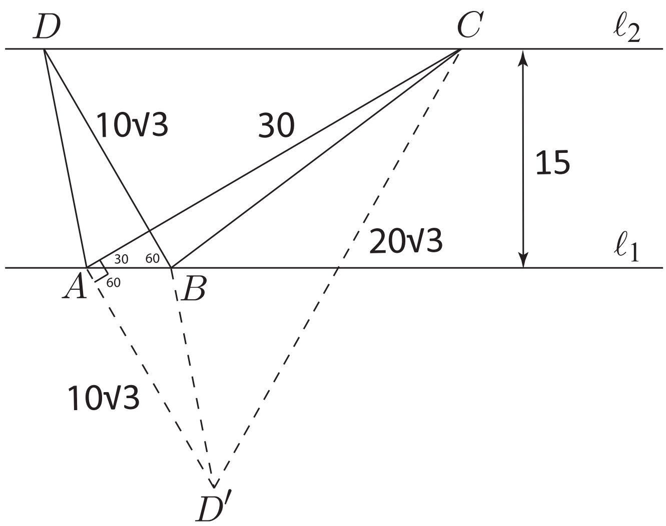

Figure 7: Problem 7 diagram.

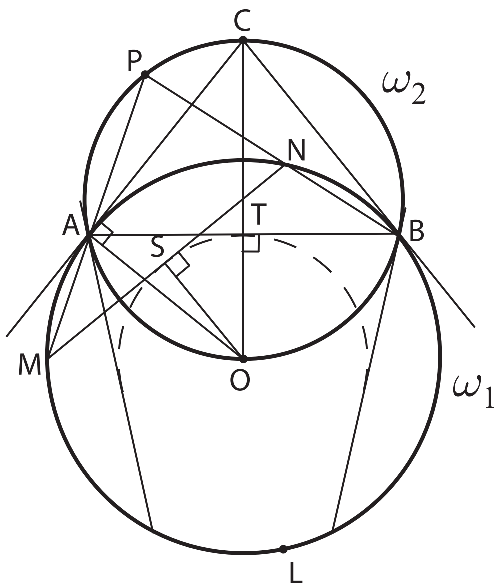

Figure 8: Problem 8 diagram.