PUMaC 2011

Combinatorics B Solutions

1. There are $\binom{5}{2} - 4 = 6$ ways to find the pair of places to put the two vowels in. Then, there are 2! ways to arrange the vowels and 3! ways to arrange the consonants. Thus, the answer is $6 \times 2! \times 3! = \boxed{72}$.

2. Since there is no carrying involved, we can do casework based on the sum's units digit. There are no sums which have a units digit of $0, 1$, or 2. If it is 3, 4, 8, or 9 then we know which two digits were added; in each of these cases there are three possible values for the sum's tens digit, after which the hundreds digit is determined. If it is 5, 6, or 7 then there are two possible pairs of added digits, and it is easily seen that in every such case there are five possible values for the sum's tens digit. Therefore there are $4 \cdot 3 + 3 \cdot 5 = \boxed{27}$ possible values for the sum.

3. Let c_n denote the number of such colorings. If the rightmost column of two squares have the same color (2 ways), then those two squares cannot be occupied by the same domino, so each must be covered by a horizontal domino. Then, the previous column must be two squares of the opposite color, and the rest of the $2 \times (n-2)$ board can be colored in c_{n-2} ways. If they have two different colors (2 ways), then one can suppose that the rightmost column is covered by a vertical domino, so that the rest of the $2 \times (n-1)$ board can be colored in c_{n-1} ways: this is only because if the rightmost column is covered by two horizontal dominos, then we can rotate those two dominos and result in a configuration where the rightmost column is covered by a vertical domino, and we still have the same coloring. Hence we have the recursion $c_n = 2c_{n-1} + 2c_{n-2}$ with $c_1 = 2, c_2 = 6$, and we compute that $c_6 = \boxed{328}$.

   Challenge: What if instead we have an infinite supply of two types of dominoes, one of which has one white square and one black square, and the other which has two black squares?

4. Since $f(1)^2 = f(1)$, then $f(1) = 1$. We have that $f(2)^3 = f(8) \leq 88, f(3)^2 = f(9) \leq 88$, so $f(2) \leq 4, f(3) \leq 9$. If $f(2) = 2$ and 3, there are respectively 5 and 6 possible values of $f(3)$, which fixes the value of each of $f(2), f(3), f(4), f(6), f(8), f(9)$. Then $f(5), f(7)$ can be any of the $81 \cdot 80$ remaining values. If $f(2) = 4$ then $f(3) \neq 1, 2, 4, 8$ still, so $f(3)$ can have 5 possible values, and it follows that the total number of such functions is $(5+6+5) \cdot 81 \cdot 80 = 2^8 \cdot 3^4 \cdot 5$, and the answer is $8 \cdot 2 + 4 \cdot 3 + 1 \cdot 5 = \boxed{33}$.

5. Fix one edge for the first point to lie on. If the second point lies on the opposite edge, it will be of distance greater than one (with 1/4 probability), and if it lies on the same edge, then it will be of distance less than one (again 1/4 probability). Suppose then that the second point lies on one of the other two adjacent edges. If the first point is distance x from this edge, then the other point must lie farther than $\sqrt{1-x^2}$ from the vertex shared by the edges the two points lie on. Therefore, this reduces to a geometric probability problem where we are finding the area of a region outside of a quarter circle in the unit square (we could also view this as the integral $\int_0^1 \sqrt{1-x^2}\,dx$, although this is a bit more work). Thus, $p = \frac{1}{4} + \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \frac{6-\pi}{8}$, and $\lfloor 100p \rfloor = \boxed{35}$.

6. To find the answer, we can instead subtract the number of ways we can position 4 bishops such that at least 3 are bishops on a diagonal from the total number of cases. Also, since the

problem asks for the remainder of the answer divided by 100, we only need to keep track of the last two digits for intermediate steps. There are in total $\binom{25}{4} \equiv 50 \pmod{100}$ ways of placing 4 bishops on the board.

• When there are 4 bishops on a diagonal, they are either on the main diagonals or the diagonals with length 4. There are

$$2\binom{5}{4} + 4 = 14$$

cases of this kind.

• Otherwise, there are exactly 3 bishops on some diagonal. They can be on the main diagonals, the diagonals with length 4, or the diagonals with length 3. The number of cases can also be calculated:

$$2\binom{5}{3}\binom{20}{1} + 4\binom{4}{3}\binom{21}{1} + 4\binom{22}{1} \equiv 0 + 16 \cdot 21 + 88 \equiv 24 \pmod{100}.$$

Therefore, we obtain the final answer to be $50 - 14 - 24 = \boxed{12}$.

7. **First solution**: Turning to generating functions, this is the same problem as asking for the coefficient of x^{263} in the polynomial

$$(1 + x + x^2 + x^3)(1 + x^2 + x^4 + x^6) \cdots (1 + x^{64} + x^{128} + x^{192})$$

and we can simplify the polynomial by telescoping:

$$\begin{aligned}\frac{x^4 - 1}{x - 1} \cdot \frac{x^8 - 1}{x^2 - 1} \cdots \frac{x^{256} - 1}{x^{64} - 1} &= \frac{x^{128} - 1}{x^2 - 1} \cdot \frac{x^{256} - 1}{x - 1} \\ &= (1 + x^2 + x^4 + \cdots + x^{126})(1 + x + x^2 + \cdots + x^{255}).\end{aligned}$$

From here we can easily calculate the coefficient of x^{263} to be $\boxed{60}$.

Second solution: First, notice the sum of all of the weights is $127 \times 3 = 381$, and any combination of weights that sum to 263 corresponds to a combination of weights that sum to $381 - 263 = 118$ (by using the leftover weights instead). So we instead consider the number of ways of forming a total weight of 118. Then, we can construct a bijection between any set of weights to the set of even nonnegative integers $2m \leq 118$: use twice the weights corresponding to the binary expansion of m, and then add the weights corresponding to the binary expansion of $118 - 2m$. This is possible because the binary expansion of any natural number less than or equal to 118 has at most 7 binary digits, corresponding to $2^6 = 64$. Conversely, for any such combination of weights, if zero or one of the 2^n weights are used, then the corresponding binary digit for m will be 0, while if two or three of the 2^n weights are used, then the corresponding binary digit for m will be 1. So again, there are $\frac{118}{2} + 1 = \boxed{60}$ ways in total.

8. Define a_i such that if in the ith minute from the beginning someone enters, then $a_i = 1$, and if someone leaves, then $a_i = -1$. Hence, each possible sequence of entries and exits is denoted by a sequence $\{a_i\}_{i=1}^{200}$ containing 100 values of 1 and 100 values of -1. Define $S_n = \sum_{i=1}^{n} a_i$ for

$0 \le n \le 200$, and $M = \max_{1 \le i \le 200} S_i$. First consider the number of possible sequences for which $M \ge 10$.

If a sequence $\{a_n\}$ has $M \ge 10$, we can find i such that $S_i = 10$, since $S_0 = 0$ and every S_i differs by 1 from the previous one. Let p be the minimum of these such i's, so that $S_p = 10$ and $S_i < 10$ for any $i < p$. Now define another sequence $\{b_i\}$ such that

$$b_i = \begin{cases} -a_i & \text{if } i \le p, \\ a_i & \text{if } i > p \end{cases}.$$

Then, this forms a bijection between all such sequences $\{a_n\}$ with $M \ge 10$ to all such sequences $\{b_i\}_{i=1}^{200}$ containing 90 values of 1 and 110 values of -1's, since it is not hard to check that this "reflection" is one-to-one and onto. (Essentially what we are doing is noting that for any random walk with 200 steps of ± 1 starting at 90, ending at 90, and going above 100, then we can reflect the sequence after the first time the sequence goes above 100, which gives a random walk starting at 90 and ending at 110.) Here, the number of sequences $\{b_i\}$ is $\binom{200}{90}$, so the number of $\{a_n\}$ with $M \ge 10$ is also $\binom{200}{90}$.

Similarly, the number of $\{a_i\}$ with $M \ge 11$ is $\binom{200}{89}$. Subtracting the two gives the number of sequences $\{a_i\}$ with $M = 10$. This exactly satisfies the requirement of the problem, since the maximum number of people at any time is $90 + 10 = 100$. Also, the number of people will not be negative, otherwise the number of people cannot reach 100 at any time. So

$$n = \binom{200}{90} - \binom{200}{89} = \frac{21 \times 200!}{90! \times 111!}.$$

The greatest power of 2 in a factorial $k!$ is given by Legendre's formula to be

$$v_2(k!) = \sum_{i=1}^{\lfloor \log_2 k \rfloor} \left\lfloor \frac{k}{2^i} \right\rfloor,$$

so calculating the powers in these three factorials, $m = v_2(200!) - v_2(90!) - v_2(111!) = 197 - 86 - 105 = \boxed{6}$.