PUMaC 2011

Combinatorics B

1. [**3**] How many ways are there to arrange the five letters P,U,M,A,C, such that the two vowels are not adjacent?

2. [**3**] Consider the sum $\overline{ab} + \overline{cde}$, where each of the letters is a distinct digit between 1 and 5. How many values are possible for this sum?

3. [**4**] A set of n dominoes, each colored with one white square and one black square, is used to cover a $2 \times n$ board of squares. For $n = 6$, how many different patterns of colors can the board have? (For $n = 2$, this number is 6.)

4. [**4**] A function $f : \{1, 2, \dots, n\} \to \{1, \dots, m\}$ is *multiplication-preserving* if $f(i)f(j) = f(ij)$ for all $1 \le i \le j \le ij \le n$, and *injective* if $f(i) = f(j)$ only when $i = j$. For $n = 9, m = 88$, the number of injective, multiplication-preserving functions is N. Find the sum of the prime factors of N, including multiplicity. (For example, if $N = 12$, the answer would be $2 + 2 + 3 = 7$.)

5. [**5**] Two points are chosen uniformly at random on the sides of a square with side length 1. If p is the probability that the distance between them is greater than 1, what is $\lfloor 100p \rfloor$? (Note: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

6. [**6**] Let N be the number of ways to place 4 bishops on a 5×5 chessboard such that no 3 are on the same diagonal. Find the remainder when N is divided by 100. (Note: the length of a diagonal on a 5×5 chessboard can be 2, 3, 4, or 5.)

7. [**7**] For every integer n from 0 to 6, we have 3 identical weights with weight 2^n. How many ways are there to form a total weight of 263 grams using only these given weights?

8. [**8**] At the start of the PUMaC opening ceremony in McCosh auditorium, the speaker counts 90 people in the audience. Every minute afterwards, either one person enters the auditorium (due to waking up late) or leaves (in order to take a dreadful math contest). The speaker observes that in this time, exactly 100 people enter the auditorium, 100 leave, and 100 was the largest audience size he saw. Find the largest integer m such that 2^m divides the number of different possible sequences of entries and exits given the above information.