PUMaC 2011

Algebra B Solutions

1. Since $\otimes(7,1,3) = \frac{7-1}{3} = 2$ and $\otimes(-3,-4,2) = \frac{2-(-4)}{-3} = -2$, our answer is

$$\otimes(2,-2,1) = \frac{2-(-2)}{1} = \boxed{4}.$$

2. **First Solution**: We write $x^2-2x+5 = (x-a)(x-b) = x^2-(a+b)x+ab$, so $ab = 5$, $a+b = 2$ (or we could apply Vieta's formulas). From these elementary symmetric polynomials, we can find all of the power sums of the roots:

$$\begin{aligned} a^2+b^2 &= (a+b)^2 - 2ab = 4 - 10 = -6 \\ a^4+b^4 &= (a^2+b^2)^2 - 2a^2b^2 = 36 - 50 = -14 \\ a^8+b^8 &= (a^4+b^4)^2 - 2a^4b^4 = 196 - 2\cdot 625 = -1054 \end{aligned}$$

Thus, the answer is $\boxed{1054}$.

Second Solution: By the quadratic equation, the roots of $x^2-2x+5 = 0$ are given by $x = \frac{2\pm\sqrt{4-4\cdot 5}}{2} = 1 \pm 2i = \sqrt{5}\cdot\left(\frac{1}{\sqrt{5}} \pm \frac{2i}{\sqrt{5}}\right)$. Note that these two roots are complex conjugates of each other. By De Moivre's formula,

$$|a^8+b^8| = 2\Re\left\{\sqrt{5}\operatorname{cis}\theta\right\}^8 = 2\cdot 5^4\cos 8\theta,$$

where $\cos\theta = 1/\sqrt{5}$. By three applications of the double-angle formula, $\cos 8\theta = 2(2(2\cos^2\theta - 1)^2-1)^2-1 = \frac{98}{625} - 1 = \frac{-527}{5^4}$, so $|a^8+b^8| = 2\cdot 527 = \boxed{1054}$.

Third Solution: As before, the two roots are $1\pm 2i$. Then, squaring three times, $(1+2i)^2 = -3+4i$, $(1+2i)^4 = -7-24i$, $(1+2i)^8 = -527+336i$. Similarly, $(1-2i)^8 = -527-336i$ (by taking the conjugate of both sides), so $|a^8+b^8| = 2\cdot 527 = \boxed{1054}$.

3. By inspection, we see that 1 is a root of this polynomial. Factoring out $(x-1)$, we have $f(x) = (x-1)(x^2-6x+10)$. Since $x^2-6x+10 = (x-3)^2+1$, then for any $x < 0$ or $x > 3$, both $|x-1|, |x^2-6x+10| \geq 2$, so their product cannot be prime. Trying directly, $f(0) = -10, f(1) = 0$ are not prime. If $x \geq 2$, we need at least one of $x-1$ and $(x-3)^2+1$ to be equal to 1, so we only need to consider the cases $x = 2$ and $x = 3$. At both of these, $f(2) = f(3) = 2$, so the sum of all distinct primes values taken on by $f(x)$ is $\boxed{2}$.

4. Substituting iz in the equation gives

$$-z^2 = f(iz + f(-z + f(-iz + f(z + f(iz + \ldots))))).$$

We then have

$$f(z-z^2) = f(z + f(iz + f(-z + f(-iz + f(z + \ldots = z^2$$

for all complex z. In particular, there exists some z such that $z^2 = z_0$. We see that $f(z-z^2) = z^2 = f(z^2)$. But f is one-to-one, so applying f^{-1} to both sides, $z - z^2 = z^2$. Thus, $z = 2z^2$, from which we get $z = 1/2$. Thus, $1/z_0 = 1/z^2 = \boxed{4}$.

5. Let $p(x) = (x-m)^k(x-n)^{6-k}$. Note that k cannot be even, as otherwise the coefficient of x^5 would be even. Hence, by symmetry, there are just two cases to check, where $k = 1$ (equivalent to $k = 5$) and $k = 3$. For $k = 1$, checking the coefficients of x^5 and x^4 respectively gives $m + 5n = -3$ and
$$-3 = 5mn + 10n^2 = 5n(m + 2n) = 5n(-3 - 3n),$$
so $5n(n + 1) = 1$ which certainly has no integral solutions. For $k = 3$, we obtain respectively $3m + 3n = -3 \implies m + n = -1$ and
$$-3 = 3m^2 + 3n^2 + 9mn = 3((m + n)^2 + mn) = 3(1 + mn) \implies mn = -2.$$
Hence, m and n are the roots to the quadratic $q(x) = (x-m)(x-n) = x^2-x-2 = (x-2)(x+1)$, so $\{m, n\} = \{-2, 1\}$. Thus, $p(x) = (x - 1)^3(x + 2)^3$, so the answer is $p(2) = 1^3 \cdot 4^3 = \boxed{64}$.

6. First, note that the possible end states of the machine are $\{4, 2, 1\}$ and $\{6, 3\}$, and that the machine will invariably halve itself at most every other operation, since when m is odd then the output $m + 3$ is even. Therefore, when operating in reverse order, the longest sequence will be the one that halves exactly every other time. Since the ending period $\{1, 2, 4\}$ is longer than $\{6, 3\}$ and obtains smaller values than 6, then $\{1, 2, 4\}$ end will result in the longer chain. Operating in reverse order, we can see that $\{1, 2, 4, 8, 5, 10, 7, 14, 11, 22, 19, 38, 35, 70, 67\}$ is the longest possible chain, and so the answer is $\boxed{67}$.

7. Rewrite the equation as $a_n - a_{n-1} = \frac{5}{6}(a_{n-1} - a_{n-2}) + \frac{10}{3}$. Define another sequence $\{b_n\}$ such that $b_n = a_{n+1} - a_n$. Thus, $b_1 = 1$ and $b_n = \frac{5}{6}b_{n-1} + \frac{10}{3}$ for $n \geq 2$, and if we define $\{c_n\}$ such that $c_n = b_n - 20$, then $c_1 = -19$ and $c_n = \frac{5}{6}c_{n-1}$ for $n \geq 2$. Now
$$\begin{aligned} a_{2011} &= a_0 + \sum_{n=1}^{2010}(a_n - a_{n-1}) = \sum_{n=1}^{2010} b_n = \sum_{n=1}^{2010}(c_n + 20) = 2010 \cdot 20 + \sum_{n=1}^{2010} c_n \\ &= \frac{-19 \cdot \left(1 - \left(\frac{5}{6}\right)^{2010}\right)}{1 - \frac{5}{6}} + 40200 \approx -6 \cdot 19 + 40200 = \boxed{40086}. \end{aligned}$$

8. Let $\zeta = e^{i\pi/3}$. Without loss of generality, let $\alpha_i = \zeta^i$ for each i from 1 to 6. Then we have $\alpha_3 = -1$ and $\alpha_6 = 1$. Therefore, the equations $f(\alpha_1, \ldots, \alpha_6) = \alpha_3 + 1 = 0$ and $g(\alpha_1, \ldots, \alpha_6) = \alpha_6 - 1 = 0$ show that α_3 and α_6 must be fixed by any such permutation.

   We also have that $\zeta + \zeta^5 = 1$ and $\zeta^2 + \zeta^4 = -1$. Therefore we can see that $f(\alpha_1, ..., \alpha_6) = \alpha_1 + \alpha_5 - 1 = 0$ and $g(\alpha_1, ..., \alpha_6) = \alpha_2 + \alpha_4 + 1 = 0$ are also polynomials of the desired form, so these polynomials must also be zero upon permutation, and therefore $(\alpha_2, \alpha_4) \to (\alpha_2, \alpha_4)$ or $(\alpha_2, \alpha_4) \to (\alpha_4, \alpha_2)$. Similarly, $(\alpha_1, \alpha_5) \to (\alpha_1, \alpha_5)$ or $(\alpha_1, \alpha_5) \to (\alpha_5, \alpha_1)$.

   Suppose α_2 and α_4 are fixed by a permutation that also swaps α_5 and α_1, and consider the polynomial $f(\alpha_1, ..., \alpha_6) = \alpha_1^2 - \alpha_2 = 0$. This polynomial permutes to $f(\alpha_{i_1}, ..., \alpha_{i_6}) = \alpha_5^2 - \alpha_2 = \zeta^{10} - \zeta^2 \neq 0$. Similarly, the permutation that fixes α_5 and α_1 but reverses α_2 and α_4 does not work due to the same polynomial as above. Finally, we need to show that the final two permutations do work. Clearly the identity permutation works. It remains to show that the permutation that fixes the roots ± 1 and swaps the pairs of roots (ζ, ζ^5) and (ζ^2, ζ^4) satisfies the conditions of the problem. This permutation is simply complex conjugation. Since we know that $P(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = 0$, we have

$$P(\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}, \overline{\alpha_4}, \overline{\alpha_5}, \overline{\alpha_6}) = \overline{P(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)} = 0.$$

and thus both of these permutations work, and the answer is $\boxed{2}$.