PUMαC - Power Round. Geometry Revisited

Stobaeus (one of Euclid's students): "But what shall I get by-learning these things?"
Euclid to his slave: "Give him three pence, since he must make gain out of what he learns."

- Euclid, Elements

As the title puts it, this is going to be a geometry power round. And as it always happens with geometry problems, every single result involves certain particular configurations that have been studied and re-studied multiple times in literature. So even though we tryed to make it as self-contained as possible, there are a few basic preliminary things you should keep in mind when thinking about these problems. Of course, you might find solutions separate from these ideas, however, it would be unfair not to give a general overview of most of these tools that we ourselves used when we found these results. In any case, feel free to skip the next section and start working on the test if you know these things or feel that such a section might bound your spectrum of ideas or anything!

1 Prerequisites

Exercise -6 (1 point). Let ABC be a triangle and let $A_1 \in BC$, $B_1 \in AC$, $C_1 \in AB$, with none of A_1, B_1, and C_1 a vertex of ABC. When we say that $A_1 \in BC$, $B_1 \in AC$, $C_1 \in AB$, we mean that A_1 is in the line passing through BC, not necessarily the line segment between them. Prove that if A_1, B_1, and C_1 are collinear, then either none or exactly two of A_1, B_1, and C_1 are in the corresponding line segments.

Solution. The (filled-in) triangle is convex, so its intersection with the line passing through A_1, B_1, C_1 is either empty or a line segment. Hence the intersection of the line and the (line segments only) triangle is either zero or two points. A point is in both the line and one extended of the sides of the triangle iff it's A_1, B_1, or C_1, so a point is in the intersection of the line and the (line segments only) triangle if it's one of A_1, B_1, and C_1 and in the corresponding line segment, so zero or two of them are.

Menelaus' theorem (triangle version). Let ABC be a triangle and let $A_1 \in BC$, $B_1 \in AC$, $C_1 \in AB$, with none of A_1, B_1, and C_1 a vertex of ABC. Then A_1, B_1, C_1 are collinear if and only if

$$\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = 1$$

and exactly one or all three lie outside the segments BC, CA, AB. //We use the convention that all segments are unoriented throughout this test.

Exercise -5 (4 points). Prove the quadrilateral one-way version of Menelaus's Theorem, stated below.

Draw diagonal A_1A_3; let M be its intersection with d. Then by triangle Menelaus,

$$\frac{M_1A_1}{M_1A_2} \cdot \frac{M_2A_2}{M_2A_3} \cdot \frac{MA_3}{MA_1} = 1$$

and

$$\frac{M_3A_3}{M_3A_4} \cdot \frac{M_4A_4}{M_4A_1} \cdot \frac{MA_1}{MA_3} = 1.$$

Multiplying these gives the desired result.

Menelaus's Theorem (quadrilateral one-way version).
Let $A_1A_2A_3A_4$ be a quadrilateral and let d be a line which intersects the sides A_1A_2, A_2A_3, A_3A_4 and A_4A_1 in the points M_1, M_2, M_3, and M_4, respectively. Then,

$$\frac{M_1A_1}{M_1A_2} \cdot \frac{M_2A_2}{M_2A_3} \cdot \frac{M_3A_3}{M_3A_4} \cdot \frac{M_4A_4}{M_4A_1} = 1.$$

Ceva's theorem. Let ABC be a triangle and let $A_1 \in BC$, $B_1 \in AC$, $C_1 \in AB$, with none of A_1, B_1, and C_1 a vertex of ABC. Then AA_1, BB_1, CC_1 are concurrent if and only if

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1$$

and exactly one or all three lie on the segments BC, CA, AB.

Exercise -4 (1 point). Let ABC be a triangle, and let A_1, B_1, C_1 be the midpoints of BC, CA, AB. Prove that AA_1, BB_1, and CC_1 intersect at one point.

Since A_1 is the midpoint of BC, $\frac{BA_1}{A_1C} = 1$. Similarly, $\frac{CB_1}{B_1A} = 1$ and $\frac{AC_1}{C_1B} = 1$, so $\frac{BA_1}{A_1C}\frac{CB_1}{B_1A}\frac{AC_1}{C_1B} = 1$. Also, all three of A_1, B_1, and C_1 lie on the line segments BC, CA, and AB, respectively, so by Ceva's theorem AA_1, BB_1, and CC_1 intersect at one point, as desired.

Desargues's theorem. Let ABC and $A'B'C'$ be two triangles. Let A'' be the intersection of BC and $B'C'$, B'' be the intersection of CA and $C'A'$, and C'' be the intersection of AB and $A'B'$, and assume all three of those intersections exist. Then the lines AA', BB', and CC' are concurrent or all parallel if and only if A'', B'', and C'' are collinear.

Desargues's Theorem is true even if, for instance, BC and $B'C'$ are parallel, provided that one interprets the "intersection" of parallel lines appropriately. We won't state cases like the following separately in the future; you may want to google "line at infinity" if you're unfamiliar with them.

Exercise -3 (2 points). Let ABC and $A'B'C'$ be two triangles. Suppose that BC and $B'C'$ are parallel, CA and $C'A'$ are parallel, and AB and $A'B'$ are parallel. Prove that the lines AA', BB', and CC' are concurrent or all parallel.

If some two of the lines are parallel, say AA' and BB', then $AA'B'B$ is a parallelogram, so $AB = A'B'$. Hence ABC and $A'B'C'$ are congruent. Translate A to A'; the same translation takes B to B' and preserves congruence, so it takes C to C', so CC' is also parallel to AA'.

If no pairs of the lines are parallel, let X be the intersection of AA' and BB', Y be the intersection of BB' and CC', and Z be the intersection of CC' and AA'. If some two of them are equal, say $X = Y$, then there's a point in all three lines, as desired, so assume XYZ is a (nondegenerate) triangle.

By similarity of XAB and $XA'B'$, $\frac{XB}{BB'}\frac{A'A}{AX} = 1$. Similarly (pun intended), $\frac{YC}{CC'}\frac{B'B}{BY} = 1$ and $\frac{ZA}{AA'}\frac{C'C}{CZ} = 1$. Multiplying these gives $\frac{ZA}{AX}\frac{XB}{BY}\frac{YC}{CZ} = 1$. Hence by Menelaus's theorem, A, B, and C are collinear, contradicting that ABC is a (nondegenerate) triangle.

Pascal's theorem. Let A, B, C, D, E, F be six points all lying on the same circle. Then, the intersections of AB and DE, of BC and EF, and of CD and FA are collinear. //Keep an open mind for degenerate cases, where some of the points are equal! (For this theorem, if two of the points are equal, then the line through them is the tangent to the circle at that point.)

Also, some projective geometry notions might come in handy at some point. We introduce below the notions of *harmonic division*, *harmonic pencil*, *pole*, *polar*, and mention some lemmas that you may want to be familiar with.

Let d be a line and A, C, B, and D four points which lie in that order on it. The quadruplet $(ACBD)$ is called a harmonic division (or just "harmonic") if and only if

$$\frac{CA}{CB} = \frac{DA}{DB}.$$

Exercise -2 (1 point). Suppose $A = (1, 0)$, $B = (5, 4)$, and $C = (4, 3)$. Find D such that $(ACBD)$ is harmonic, E such that $(ACEB)$ is harmonic, and F such that $(AFCB)$ is harmonic.

Answer: $D = (7, 8)$, $E = (\frac{31}{7}, \frac{38}{7})$, $F = (\frac{17}{5}, \frac{22}{5})$: check the definitions.

If X is a point not lying on d, then the "pencil" $X(ACBD)$ (consisting of the four lines XA, XB, XC, XD) is called harmonic if and only if the quadruplet $(ACBD)$ is harmonic. So in this case, $X(ACBD)$ is usually called a *harmonic pencil*.

Now, the polar of a point P in the plane of a given circle Γ with center O is defined as the locus (that is, set) of points Q such that the quadruplet $(PXQY)$ (or $(QXPY)$, if P is inside the circle) is harmonic, where XY is the chord of Γ passing through P. This polar is actually a line perpendicular to OP; when P lies outside the circle, it is precisely the line determined by the tangency points of the circle with the tangents from P. We say that P is the pole of this line with respect to Γ.

Exercise -1 (1 point). Let O be the circle of radius 2 centered at the origin, and let $P = (1, 0)$. Find the polar of P.

Answer: $y = 4$: check the definition.

Given these concepts, there are four important lemmas that summarize the whole theory. Remember them by heart even though you might prefer not to use them in this test. They are quite beautiful.

Lemma 1. In a triangle ABC consider three points X, Y, Z on the sides BC, CA, and AB, respectively. If X' is the point of intersection of the line YZ with the extended side BC, then the quadruplet $(BXCX')$ is a harmonic division if and only if the lines AX, BY, CZ are concurrent.

Lemma 2. Let A, B, C, D be four points lying in this order on a line d, and let X be a point not lying on this line. Take another line d' and consider its intersections A', B', C', D' with the lines XA, XB, XC, and XD, respectively. Then, the quadruplet $(ABCD)$ is harmonic if and only if $(A'B'C'D')$ is harmonic.

Conversely, if $A = A'$ and both $(ABCD)$ and $(A'B'C'D')$ are harmonic, the lines BB', CC', DD' are concurrent.

Lemma 3. Let A, B, C, D be four points lying in this order on a line d. If X is a point not lying on this line, then if two of the following three propositions are true, then the third is also true:

- The quadruplet $(ABCD)$ is harmonic.
- XB is the internal angle bisector of $\angle AXC$.
- $XB \perp XD$.

Lemma 4. If P lies on the polar of Q with respect to some circle Γ, then Q lies on the polar of P.

Exercise 0 (

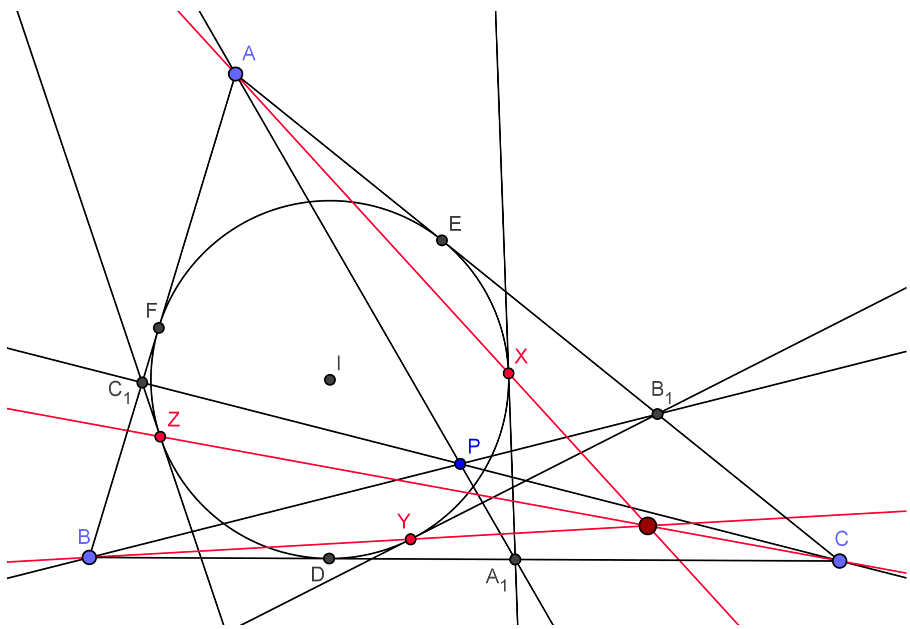
This is it. You are now ready to proceed the next section. As Erdős would advise you, keep your brain open!

2 Solutions.

General Setting. We are initially given a scalene triangle ABC in plane, with incircle Γ, for which we denote by D, E, F the tangency points of Γ with the sides BC, CA, and AB, respectively. Furthermore, let P be an arbitrary point in the *interior*[1] of the triangle ABC and let $A_1B_1C_1$ its cevian triangle (that is, A_1, B_1,

[1] The point P is considered inside ABC just for convenience, so please don't worry about the word *interior* that much; we just want nice, symmetrical drawings!)

C_1 are the intersections of the cevians AP, BP, CP with the sidelinse BC, CA, and AB, respectively); from A_1, B_1, C_1 draw the tangents to Γ that are different from the triangle's sides BC, CA, AB, and take their tangency points with the incircle to be X, Y, and Z, respectively. Now, with this picture in front of you, take a look at the following results! (ehm... and prove them!)



Proposition 1 (a) (3 points). Extend A_1X to intersect AC at V. Show that the lines AA_1, BV, and DE are concurrent. [This is a general fact called **Newton's Theorem**; you might want to remember this for the rest of the test!]

(b) (6 points). Let A_2 be the intersection point of AX with the sideline BC. Show that

$$\frac{A_2B}{A_2C} = \frac{s-c}{s-b} \cdot \frac{A_1B^2}{A_1C^2}.$$

[Hint: Menelaus, Menelaus, Menelaus!]

(c) (1 point). Prove that the lines AX, BY, and CZ are concurrent.

Solution. (a) We are dealing with the following configuration:

Lemma (**Newton's theorem**). Let $ABCD$ be a circumscriptible quadrilateral with inscribed circle Γ. Suppose X, Y, Z, W are the tangency points of Γ with AB, BC, CD, and DA, respectively. Then, the lines AC, BD, XZ, and YW are concurrent.

Proof of Lemma. Let P be the point of intersection of the lines AC and YW. The lines BC and DA touch the incircle of the quadrilateral $ABCD$ at the points

Y and W; hence both angles $\angle CYW$ and $\angle DWY$ are equal to the chordal angle of the chord YW in Γ. Thus, $\angle CYW = \angle DWY$, and in other words, $\angle CYP = 180° - \angle AWP$. Thus, $\sin CYP = \sin AWP$. But after the Law of Sines in triangle AWP, we have $AP = AW \cdot \frac{\sin AWP}{\sin APW}$, and after the Law of Sines in triangle CYP, we have $CP = CY \cdot \frac{\sin CYP}{\sin CPY}$. Thus,

$$\frac{AP}{CP} = \frac{AW \cdot \frac{\sin AWP}{\sin APW}}{CY \cdot \frac{\sin CYP}{\sin CPY}} = \frac{AW \cdot \frac{\sin AWP}{\sin APW}}{CY \cdot \frac{\sin AWP}{\sin APW}} = \frac{AW}{CY} = \frac{a}{c}.$$

Now, let P' be the point of intersection of the lines AC and XZ. Then, we similarly find $\frac{AP'}{CP'} = \frac{a}{c}$. Thus, $\frac{AP}{CP} = \frac{AP'}{CP'}$. This means that the points P and P' divide the segment AC in the same ratio; hence, the points P and P' coincide; it follows that the lines AC, XZ and YW are concurrent. Similarly, we can verify that the lines BD, XZ and YW are concurrent; therefore, all four lines AC, BD, XZ and YW are concurrent. This proves Newton's theorem.

Part (a) follows by applying the Lemma to the circumscriptible quadrilateral ABA_1V.

(b) We claim that

$$\frac{A_2B}{A_2C} = \frac{s-c}{s-b} \cdot \left(\frac{A_1B}{A_1C}\right)^2.$$

Denote A_1B/A_1C by m and let V be the intersection of A_1X and AC. We have that $A_1D = A_1X$ and $VE = VX$.

On the other hand, Newton's theorem in the circumscriptible quadrilateral ABA_1V says that the lines AA_1, BV and DE are concurrent at a point S.

WLOG, assume that $A_1 \in (DC)$. We write

$$A_1D = A_1B - BD = \frac{ma}{1+m} - (s-b) \;\text{ so }\; DA_1 = \frac{m(s-c)-(s-b)}{1+m}.$$

By Menelaus' theorem applied in triangle AA_1C for $\overline{DSE}$,

$$\frac{DA_1}{DC} \cdot \frac{EC}{EA} \cdot \frac{SA}{SA_1} = 1 \text{ i.e. } \frac{m(s-c)-(s-b)}{(1+m)(s-c)} \cdot \frac{s-c}{s-a} \cdot \frac{SA}{SA_1} = 1,$$

so

$$\frac{SA}{SA_1} = \frac{(1+m)(s-a)}{m(s-c)-(s-b)}.$$

By Menelaus' theorem applied in the same triangle AA_1C for $\overline{BSV}$,

$$\frac{BA_1}{BC} \cdot \frac{VC}{VA} \cdot \frac{SA}{SA_1} = 1 \text{ thus } \frac{m}{1+m} \cdot \frac{VC}{VA} \cdot \frac{(1+m)(s-a)}{m(s-c)-(s-b)} = 1.$$

Hence,

$$\frac{VA}{m(s-a)} = \frac{VC}{m(s-c)-(s-b)} = \frac{b}{mb-(s-b)}.$$

But $EV = AV - AE = AV - (s-a)$, and so using (3), we get

$$EV = AV - AE = \frac{m(s-a)b}{mb-(s-b)} - (s-a) \text{ and so } EV = \frac{(s-a)(s-b)}{mb-(s-b)}.$$

Now, by Menelaus' theorem applied in triangle A_1VC for $\overline{AXA_2}$, we have

$$\frac{AV}{AC} \cdot \frac{A_2C}{A_2A_1} \cdot \frac{XA_1}{XV} = 1,$$

and so

$$\frac{m(s-a)}{mb-(s-b)} \cdot \frac{A_2C}{A_2A_1} \cdot \frac{A_1D}{VE} = 1,$$

i.e.

$$\frac{m(s-a)}{mb-(s-b)} \cdot \frac{A_2C}{A_2A_1} \cdot \frac{m(s-c)-(s-b)}{1+m} \cdot \frac{mb-(s-b)}{(s-a)(s-b)} = 1.$$

Therefore,

$$\frac{A_2A_1}{m\left[m(s-c)-(s-b)\right]} = \frac{A_2C}{(1+m)(s-b)} = \frac{a}{(1+m)\left[m^2(s-c)+(s-b)\right]}.$$

It follows that

$$A_2C = \frac{a(s-b)}{m^2(s-c)+(s-b)}, \text{ thus } A_2B = BC - A_2C = \frac{m^2(s-c)a}{m^2(s-c)+(s-b)},$$

so

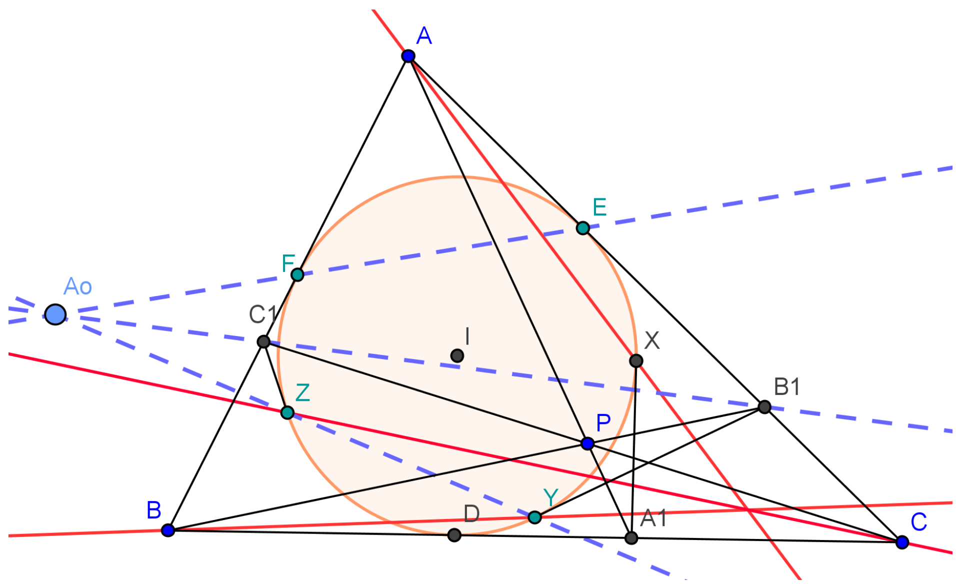
$$\frac{A_2B}{A_2C} = \frac{s-c}{s-b} \cdot m^2.$$

This proves (b).

(c) It suffices to notice that the relation from (b) is cyclic, so we actually get

$$\prod_{cyc}\left(\frac{A_2B}{A_2C}\right) = \prod_{cyc}\left(\frac{s-c}{s-b}\right) \cdot \prod_{cyc}\left(\frac{A_1B}{A_1C}\right)^2 = \prod_{cyc}\left(\frac{A_1B}{A_1C}\right)^2 = 1,$$

and so Ceva's theorem yields the concurrency of the lines AX, BY, and CZ. This proves Proposition 1. □



Proposition 2 (7 points). The lines $B'C'$, EF, YZ are concurrent at a point, say A_0. Similarly, the lines $C'A'$, FD, ZX and $A'B'$, DE, XY concur at points B_0 and C_0, respectively.

Solution. Let A_0 be the point of intersection point of EF and YZ. The concurrency is thus equivalent with the collinearity of the points A_0, C_1, B_1.

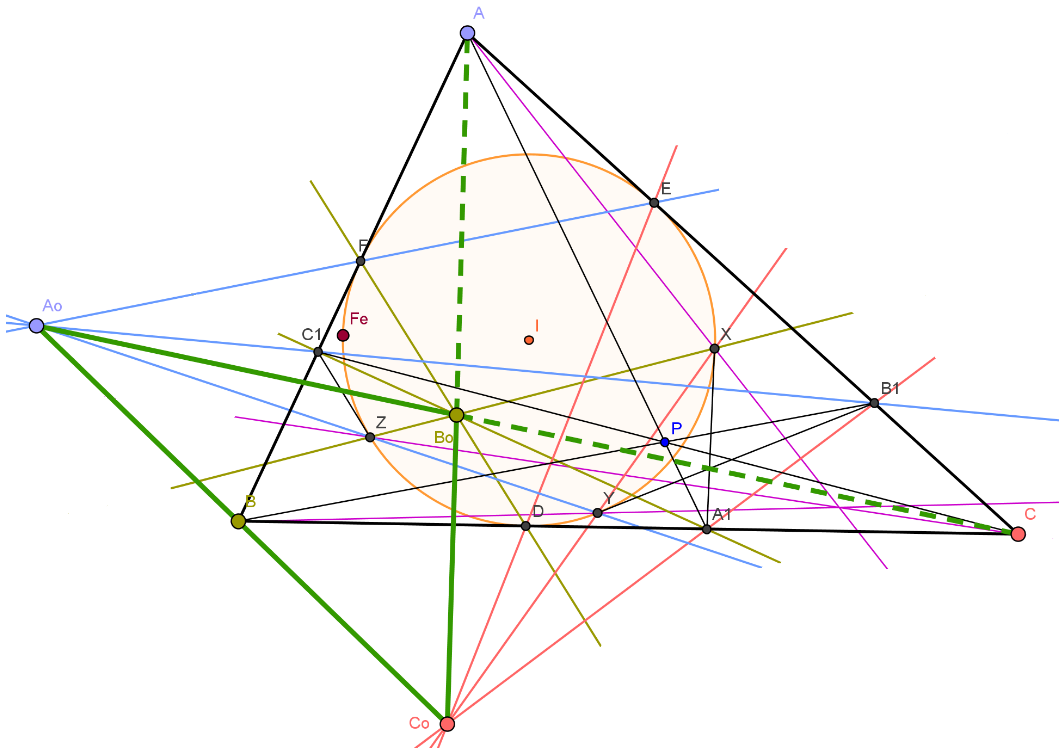
By Pascal's theorem applied to $FFEZZY$, the points C_1, A_0 and $EZ \cap YF$ are collinear. On other hand, the same Pascal theorem this time applied to $EEFYYZ$, says that the points B_1, A_0 and $FY \cap ZE$ are collinear. Therefore, it follows that A_0, C_1, and B_1 are collinear, and so Proposition 2 is proven. $\square$

Proposition 3 (a) (3 points) Let ℓ, ℓ' be two lines intersecting at P. Let X, Y, Z be points on ℓ and X', Y', Z' points on ℓ' such that

$$\frac{YX}{YZ} : \frac{PX}{PZ} = \frac{Y'X'}{Y'Z'} : \frac{PX'}{PZ'}.$$

Then, the lines XX, YY' and ZZ' are concurrent. //When the points X, Y, Z and X', Y', Z' satisfy the above equality, we also say that the cross-ratios (X, Y, Z, P) and (X', Y', Z', P') are equal.

(b) (4 points). The points A_0, B, C_0 are collinear. Similarly, the points A_0, B_0, C and A, B_0, C_0 are collinear.



Solution. (a) Just generalize your idea from Lemma 2.

(b) Suppose the lines AC and A_0C_0 intersect at K. By (a), it suffices to show that the cross-ratios (A, F, B, C_1) and (A, E, K, B_1) are equal. According to Ceva's theorem,

$$\frac{AF}{FB} = \frac{AE}{EC} \cdot \frac{CD}{DB} \text{ and } \frac{AC_1}{C_1B} = \frac{AB_1}{B_1C} \cdot \frac{CA_1}{A_1B}.$$

Since the cross-ratios (C, D, B, A_1) and (C, E, K, B_1) are equal, it follows that

$$\frac{AF}{FB} : \frac{AC_1}{C_1B} = \frac{AE}{EC} \cdot \frac{CD}{DB} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = \frac{AE}{EC} \cdot \frac{CE}{EK} \cdot \frac{B_1K}{B_1C} \cdot \frac{CB_1}{B_1A} = \frac{AE}{EK} : \frac{AB_1}{B_1K}.$$

This means that the cross-ratios (A, F, B, C_1) and (A, E, K, B_1) are indeed equal, which completes the proof. □

Proposition 4 (7 points). The triangles $A_0B_0C_0$ and DEF are *perspective*[2] and the locus of the *perspector*[3] is the incircle of triangle ABC.

[2]Two triangles ABC and XYZ are said to be perspective if and only the lines AX, BY, CZ are concurrent

[3]The perspector of two perspective triangles ABC and XYZ is precisely the common point of the lines AX, BY, CZ.

Solution. Let Q be the second intersection of the line A_0D with the incircle. By Pascal's theorem applied to $QDDEFF$, the points A_0, B, $DE \cap FQ$ are collinear; hence, the lines A_0B, FQ, and ED are concurrent.

Since $\{C_0\} = ED \cap A_0B$, it follows that Q lies on C_0F. Similarly, by Pascal's theorem applied to $FQEEDF$, Q lies on B_0E; hence, the lines A_0D, B_0E, and C_0F are concurrent on the incircle at Q. This proves Proposition 4. □

Proposition 5 (a) (5 points). Let ABC be an arbitrary triangle, and A', B', C' be three points on its sides BC, CA, AB. Let also A", B", C" be three points on the sides $B'C'$, $C'A'$, $A'B'$ of triangle $A'B'C'$. Then consider the following three assertions:

(1) The lines AA', BB', CC' concur.

(2) The lines $A'A''$, $B'B''$, $C'C''$ concur.

(3) The lines AA'', BB'', CC'' concur.

Then, if any two of these three assertions are valid, then the third one must hold, too. [Hint: We didn't put the quadrilateral version of Menelaus in the Prerequisites section for nothing! //Btw, this is called the **Cevian Nest Theorem**; remember it, it might prove useful!]

(b) (2 points). The triangles $A_0B_0C_0$ and ABC are perspective.

Solution. (a) Let the lines AA", BB", CC" meet the sides BC, CA, AB of triangle ABC at the points X, Y, Z, respectively. Of course, the lines AA", BB", CC" are nothing but the lines AX, BY, CZ.

Apply the Menelaus theorem for quadrilaterals to the quadrilateral $BCB'C'$ with the collinear points X, A, A", A on its sides (you are reading right - the point A occurs twice, it is actually the point of intersection of two opposite sides of the quadrilateral!). Then, you obtain

$$\frac{BX}{XC} \cdot \frac{CA}{AB'} \cdot \frac{B'A''}{A''C'} \cdot \frac{C'A}{AB} = 1,$$

so that

$$\begin{aligned}\frac{BX}{XC} &= \frac{AB}{C'A} \cdot \frac{A''C'}{B'A''} \cdot \frac{AB'}{CA} \\ &= \frac{AB'}{C'A} \cdot \frac{AB}{CA} \cdot \frac{A''C'}{B'A''}.\end{aligned}$$

Similarly,

$$\frac{CY}{YA} = \frac{BC'}{A'B} \cdot \frac{BC}{AB} \cdot \frac{B''A'}{C'B''} \text{ and } \frac{AZ}{ZB} = \frac{CA'}{B'C} \cdot \frac{CA}{BC} \cdot \frac{C''B'}{A'C''}.$$

Hence,

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \left(\frac{AB'}{C'A} \cdot \frac{BC'}{A'B} \cdot \frac{CA'}{B'C}\right)\left(\frac{A''C'}{B'A''} \cdot \frac{B''A'}{C'B''} \cdot \frac{C''B'}{A'C''}\right),$$

i.e. $w = uv$, where $w = \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB}$, $u = \frac{AB'}{C'A} \cdot \frac{BC'}{A'B} \cdot \frac{CA'}{B'C}$, and $v = \frac{A''C'}{B'A''} \cdot \frac{B''A'}{C'B''} \cdot \frac{C''B'}{A'C''}$.

But by Ceva's theorem assertion (1) holds iff $u = 1$, (2) holds iff $v = 1$, and (3) holds iff $w = 1$, therefore the relation $w = uv$ proves (a).

(b) This now follows by Part (a) and Proposition 4. □

Proposition 6 (a) (2 points) Let l and l' be two lines in plane and let P be a point in the same plane but not lying on either l or l'. Let A, B, C, D be points on l and let A', B', C', D' be the intersections of the lines PA, PB, PC, PD with l', respectively. Then,

$$\frac{BA}{BC} : \frac{DA}{DC} = \frac{B'A'}{B'C'} : \frac{D'A'}{D'C'}.$$

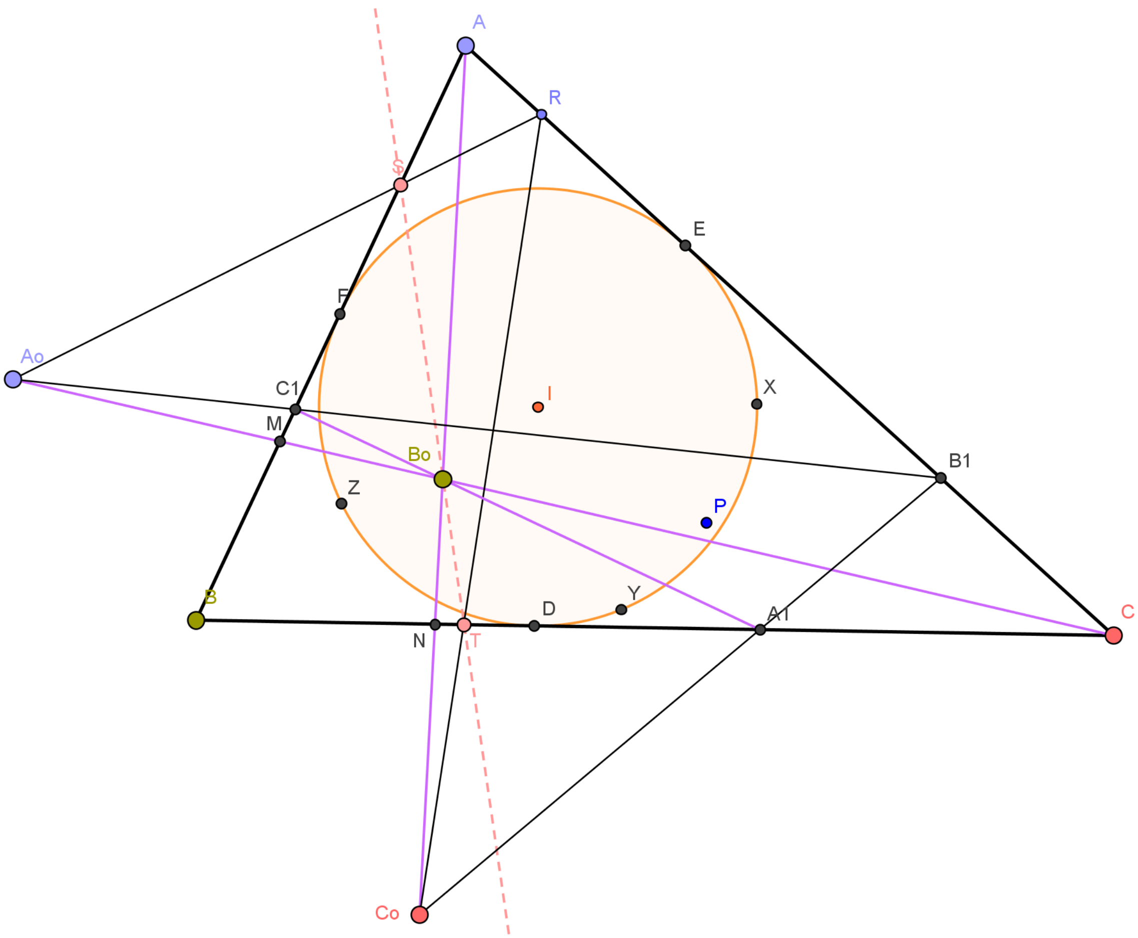
(b) (5 points). The incenter I is the orthocenter of triangle $A_0B_0C_0$.

Solution. (a) Imitate the proof you (should) have for Lemma 2 from the Prerequisites section. If your proof is ugly, then try thinking about a new solution using only the Law of Sines!

(b) Since $A_1B_1C_1$ is a cevian triangle, the pencil $(B_0A_1,\ B_0C,\ B_0B_1,\ B_0A)$ is harmonic. Thus, by using the collinearity of A, B_0 and C_0, the pencil $(B_0C_1,\ B_0A_0,\ B_0B_1,\ B_0A)$ is also harmonic.

Hence, B_0 lies on the polar of A_0 with respect to the incircle, and since A lies too it follows that B_0C_0 is the polar of A_0 with respect to the incircle.

Similarly, C_0A_0 is the polar of B_0 and A_0B_0 the polar of C_0. Therefore, I is indeed the orthocenter of triangle $A_0B_0C_0$. □

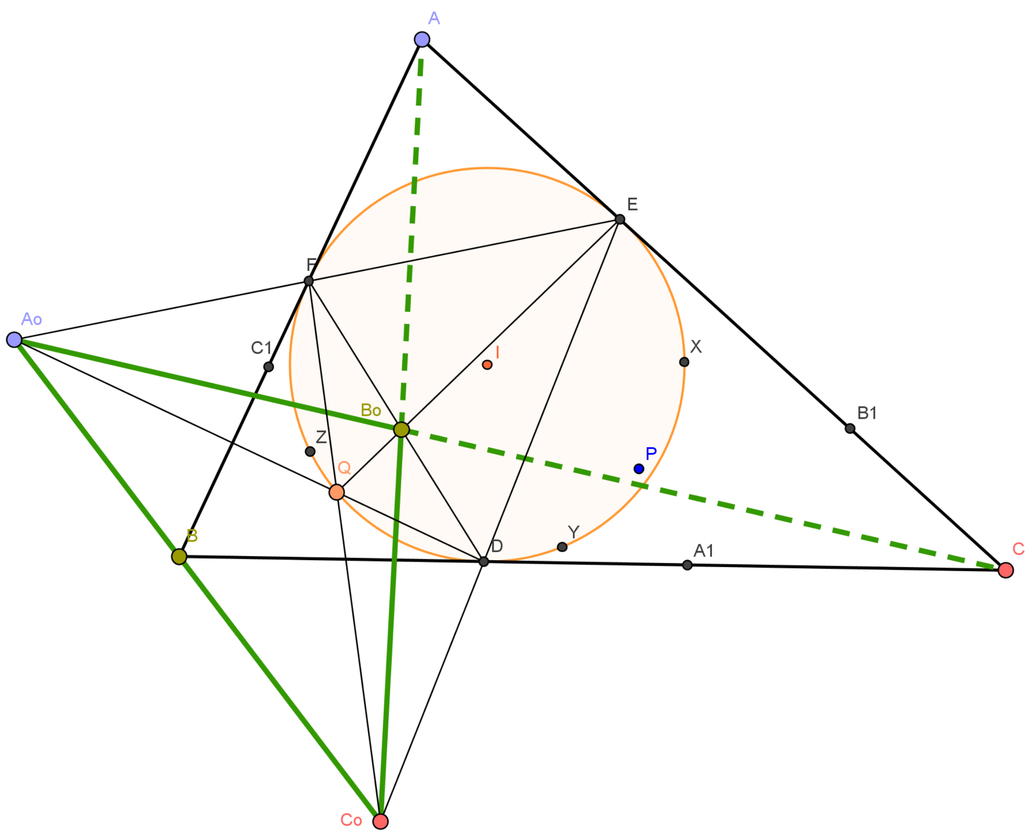


Proposition 7 (7 points). Let R a mobile point on AC and consider S, T the intersections of RA_0, RC_0 with AB and BC respectively. Then, B_0 lies on ST.

Solution. Let M and N be the intersections of B_0A_0 and B_0C_0 with the sides BA and BC, respectively.

Since (C_1, A_1), (S, T), (M, N) are the traces left by the lines (B_1A_0, B_1C_0), (RA_0, RC_0) and (CA_0, AC_0) on the sides BA and BC, it follows that the cross-ratios (A, S, C_1, M) and (N, T, A_1, C) are equal.

Hence, the lines AN, ST, C_1A_1, MC are concurrent. But $\{B_0\} = AN \cap C_1A_1 \cap MC$, thus B_0 lies on ST, which completes the proof of Proposition 7. $\square$



Proposition 8 (7 points). The triangles ABC and TRS are perspective.

Solution 1. Since $\overline{AB_0C_0}$ is the polar of A_0 with respect to $\angle BAC$, the pencil $(B_0R,\ B_0A,\ B_0S,\ B_0A_0)$ is harmonic. Thus, by the collinearity of S, B_0, and T, it follows that the pencil $(TS,\ TA,\ TR,\ TC)$ is harmonic. Hence, the lines AT, BR, CS concur, i.e. the triangles ABC and TRS are perspective. This proves Proposition 8. □

Solution 2. Recall that A_0, E, F, B_0, D, F, C_0, D, E are collinear.

We apply Menelaus's theorem to points A_0SR on $\triangle AEF$,

$$\frac{EA_0}{A_0F} \cdot \frac{FS}{SA} \cdot \frac{AR}{RE} = 1$$

to points B_0ST on $\triangle BDF$,

$$\frac{FB_0}{B_0D} \cdot \frac{DT}{TB} \cdot \frac{BS}{SF} = 1$$

to points C_0RT on $\triangle CDE$,

$$\frac{DC_0}{C_0E}\cdot\frac{ER}{RS}\cdot\frac{CT}{TD}=1$$

and finally apply Ceva's theorem to the perspective triangles $A_0B_0C_0$ and DEF:

$$\frac{FA_0}{A_0E}\cdot\frac{DB_0}{B_0F}\cdot\frac{EC_0}{C_0D}=1.$$

Multiplying all these together, we get that

$$\frac{BS}{SA}\cdot\frac{CT}{TB}\cdot\frac{AR}{RC}=1$$

which is Ceva's theorem for points TRS in $\triangle ABC$, which shows that the two triangles are perspective, as desired. □

Proposition 9 (14 points). The triangles $A_0B_0C_0$ and TRS are perspective and the locus of the perspector is a line tangent to the incircle of triangle ABC. (Hint: everything used so far is part of the hint!)

Solution. Let M and N be the intersections of B_0C_0 and A_0B_0 with the sidelines BC and AB, respectively. Also, let K, L be the intersections of MN with RS and RT, respectively.

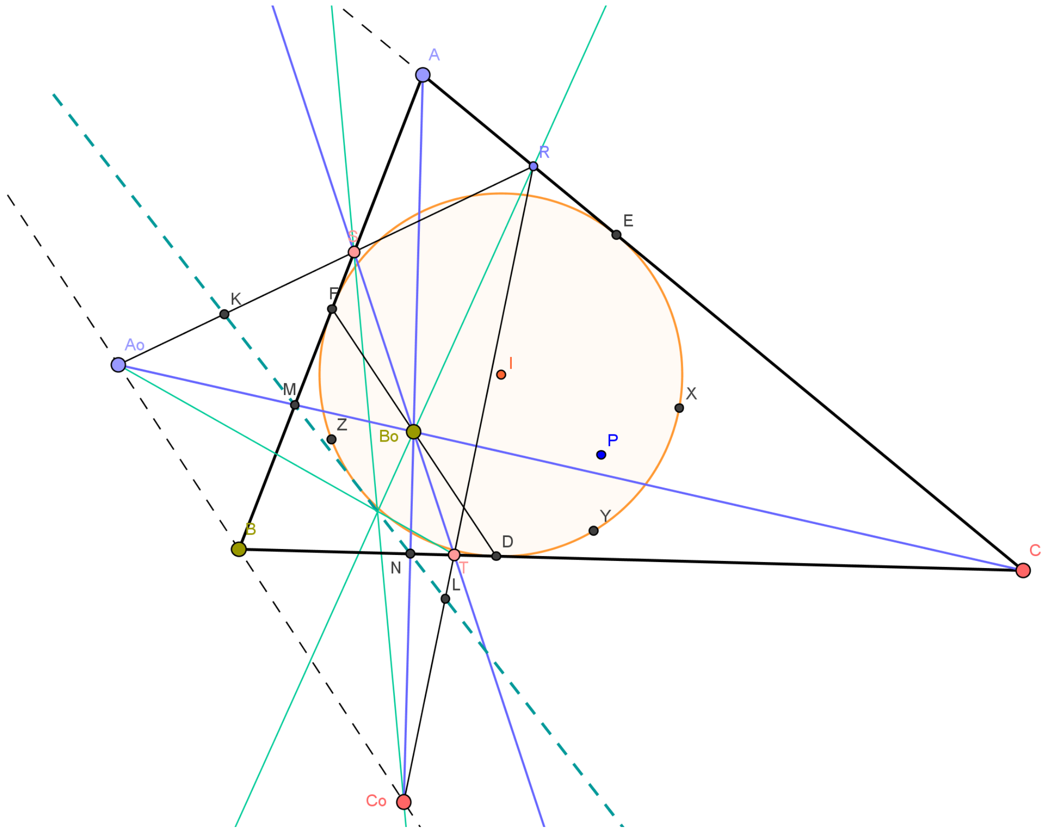
First, note that triangles $A_0B_0C_0$ and TRS are perspective by the Cevian Nest Theorem. Since the triangles ABC and $A_0B_0C_0$ are perspective, the points $AA_0\cap CC_0$, $A_0M\cap C_0N$, $AM\cap CN$ lie on the same line, i.e. the triangles AA_0M and CC_0N are perspective; consequently, the lines AC, A_0C_0, MN concur at a point, say O'.

However the lines AN, ST, MC are concurrent, so the cross-ratios (B, A, S, M) and (B, N, T, C) are equal. By intersecting the pencil $O'B$, $O'A$, $O'S$, $O'M$ with the line RS, it follows that $(B, A, S, M)=(A_0, R, S, K)$. Similarly, by intersecting the pencil $O'B$, $O'N$, $O'T$, $O'C$ with the line RT, $(B, N, T, C)=(C_0, L, T, R)$. Therefore, $(A_0, R, S, K)=(C_0, L, T, R)$, i.e. $(A_0, R, S, K)=(T, R, C_0, L)$.

Hence, the lines A_0T, SC_0, KL are concurrent, that is the lines A_0T, B_0R, C_0S, MN are concurrent. From the above result, we deduce that the locus of the perspector of $A_0B_0C_0$ and TRS is the line MN.

By Proposition 4, the lines A_0D, B_0E, C_0F concur on the incircle at Q. Consider M' and N' the intersections of the tangent in Q with the sides BA and BC, respectively. By Newton's theorem applied to the circumscribed quadrilateral $M'N'CA$, the lines QE, DF, $M'C$, $N'A$ are concurrent at B_0.

Hence, $M'\equiv M$ and $N'\equiv N$. Therefore, the line MN is tangent to the incircle at Q. This completes the proof of Proposition 9. □



3 Historical Fact/Challenge

When the point P chosen at the beginning of the excursion is the incenter, the Nagel point or the orthocenter of triangle ABC, the configuration generates a so-called Feuerbach family of circles. More precisely, maintaining the notations from the previous propositions, the circumcircle of triangle TRS always passes through the *Feuerbach point*[4] of triangle ABC, as R varies on the line AC. So, do try and have fun with it after finishing everything. You will *not* receive any credit of course, but you definitely have the chance to impress!

[4] You should be aware that the incircle and the nine-point circle of a given triangle ABC are tangent to each other and that their tangency point is called the Feuerbach point of triangle ABC!