



Number Theory A Solutions

1. The sum of the divisors of $n = 2^i 3^j$ is equal to $(1+2^1+2^2+\dots+2^i)(1+3^1+3^2+\dots+3^j) = 1815$, since each divisor of $2^i 3^j$ is represented exactly once in the sum that results when the product is expanded. Let $A = 1 + 2^1 + 2^2 + \dots + 2^i = 2^{i+1} - 1$ and $B = 1 + 3^1 + 3^2 + \dots + 3^j$, so that $AB = 1815 = 3 \cdot 5 \cdot 11^2$.

Since $B \equiv 1 \pmod{3}$, $3|A$. By Fermat's Little Theorem, $2^{i+1} - 1 \equiv 0 \pmod{3}$ only when i is odd. For $i = 1$ we get $A = 3, B = 605$ which does not work. For $i = 3$ we get $A = 15, B = 121$, which holds for $j = 4$ and $n = \boxed{648}$. For $i = 5, 7$, and 9 , we obtain $7|A, 17|A$, and $31|A$ respectively (all of which do not divide 1815), and for $i > 10$, $A > 1815$.

2. Using the identity that $\text{lcm}(m, n) \cdot \text{gcd}(m, n) = m \cdot n$, it follows that

$$3m \times \text{gcd}(m, n) = \text{lcm}(m, n) = \frac{m \cdot n}{\text{gcd}(m, n)} \implies n = 3[\text{gcd}(m, n)]^2.$$

It follows that n must be three times a perfect square. If we set $m = \sqrt{n/3}$, which is an integer, it follows that

$$\text{lcm}(\sqrt{n/3}, n) = n = 3\sqrt{n/3} \cdot \sqrt{n/3} = 3\sqrt{n/3} \cdot \text{gcd}(\sqrt{n/3}, n),$$

as desired. Hence, every triple of a perfect square works as a value of n , and the largest such under 1000 is $3 \cdot 18^2 = \boxed{972}$.

3. Note that $7^3 = 343 \equiv -1 \pmod{43}$ and that $6^6 = (6^3)^2 \equiv 1 \pmod{43}$. Therefore, for $p \equiv 0, 1, 2, 3, 4, 5 \pmod{6}$, $7^p - 6^p + 2 \equiv 2, 3, 15, 0, 32, 3 \pmod{43}$. Therefore, if $43|7^p - 6^p + 2$, $p \equiv 3 \pmod{6}$. This means that $p = \boxed{3}$ is the only solution, so that the sum of all solutions is 3.

4. Let the set be $\{a, b, c\}$, and without loss of generality, suppose that $a \leq b \leq c$. Then

$$abc - 2a - 2b - 2c = 4.$$

If $c \geq b \geq a \geq 4$, then

$$4 = abc - 2a - 2b - 2c \geq 16c - 2c - 2c - 2c = 10c \geq 40,$$

which is a contradiction. Thus, $a \in \{1, 2, 3\}$.

If $a = 1$, we get that $bc - 2b - 2c = 6$. Completing the rectangle, $bc - 2b - 2c + 4 = (b-2)(c-2) = 10$. Thus, $b-2$ and $c-2$ are a pair of (positive) factors of 10, and so must be equal to 1, 10 or 2, 5. Thus, we get the solutions $\{a, b, c\} = \{1, 3, 12\}$ and $\{1, 4, 7\}$, which can be permuted in 12 ways (6 for each solution).

If $a = 2$, we get that $2bc - 2b - 2c = 8$, so $bc - b - c + 1 = (b-1)(c-1) = 5$. Thus, $b-1$ and $c-1$ are a pair of factors of 5, and so must be equal to 1, 5. This gives the solution $\{a, b, c\} = \{2, 2, 6\}$, which can be permuted in 3 ways.



If $a = 3$, we get that $3bc - 2b - 2c = 10$, so $9bc - 6b - 6c + 4 = (3b - 2)(3c - 2) = 34$. Then $3b - 2, 3c - 2$ are a pair of factors of 34 and $(3b - 2, 3c - 2) = (1, 34), (2, 17)$. In the first pair of solutions, we have $b = 1 \leq a = 3$, and in the second pair of solutions, we do not get integer values for b, c .

In total, we have $\boxed{15}$ ordered triplets which satisfy the given conditions.

5. First, show that this sum converges by showing that $d(n) \leq 2\sqrt{n}$ for all $n \geq 1$. Let x be some divisor of n with $x \leq \sqrt{n}$. There is a one to one correspondence between divisors at most \sqrt{n} and divisors at least \sqrt{n} (map x to $\frac{n}{x}$ for any $x \leq n$). However, there are at most \sqrt{n} divisors of n that are at most \sqrt{n} , so $d(n) \leq 2\sqrt{n}$ for all positive integers n . Therefore, the given sum is bounded above by

$$\sum_{n=1}^{\infty} \frac{2}{x^{\frac{3}{2}}}$$

which converges.

First solution: Note that $d(n) = \sum_{k|n} 1$, so $\frac{d(n)}{n^2} = \sum_{k|n} \frac{1}{k^2(n/k)^2}$. This means that the desired sum is equal to

$$\sum_{n=1}^{\infty} \sum_{k|n} \frac{1}{k^2(n/k)^2} = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{k^2 m^2}.$$

This can be seen by letting $m = \frac{n}{k}$ and switching the order of summation (which is valid by the absolute convergence of the original series). Therefore, we can write this series as

$$\left(\sum_{m=1}^{\infty} \frac{1}{m^2} \right)^2 = \frac{\pi^4}{36}.$$

Hence $p(x) = \frac{x^4}{36}$ and $p(6) = \boxed{36}$. $p(x)$ is unique because π is transcendental.

Second solution: Since the original sum converges, one may use the unique prime factorization of the integers and the fact that $d(n)$ is multiplicative to factor the given sum as

$$\prod_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{d(p_i^j)}{p_i^{2j}}$$

where p_i is the i th prime. Note also that $d(p^j) = j + 1$ for any prime p . Now, compute the sum

$$S(x) = \sum_{j=0}^{\infty} (j+1)x^j$$

To compute a closed form for $S(x)$, consider $xS(x) = \sum_{j=1}^{\infty} jx^j$ and $S(x) - xS(x) = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$ for $|x| < 1$. Therefore, $S(x) = \frac{1}{(1-x)^2} = \left(\sum_{j=0}^{\infty} x^j \right)^2$. Letting $x = \frac{1}{p_i}$ for each term of the



infinite product shows that the desired sum is equal to $(\sum_{j=1}^{\infty} \frac{1}{j^2})^2 = \frac{\pi^4}{36}$. Hence $p(x) = \frac{x^4}{36}$ and $p(6) = \boxed{36}$.

6. In order to decrease the number of remainders $(\text{mod } b)$ that can be written in the form $x^3 + y^4$, we should minimize the number of cubes and fourth powers $(\text{mod } b)$. This happens when b is prime and when $3, 4 \mid b - 1$, which happens when $b = 13$. Indeed, for this value of b , the cubic residues are 0, 1, 5, 8 and 12 and the quartic residues are 0, 1, 3, and 9. 7 cannot be written as the sum of any cubic-quartic residue pair, so $(7, 13)$ satisfies the problem's constraints.

Now, I will show that for any $b < 13$, all residues $(\text{mod } b)$ can be written as a sum of a cubic-quartic residue pair. We can reduce the checking to just prime values of b and powers of primes by the Chinese Remainder Theorem. Therefore, we just need to check $b = 1, 2, 3, 4, 5, 7, 8, 9, 11$. 1, 2, and 3 are trivial to check, so just look at 4, 5, 7, 8, 9, 11. For 5, note that $x^4 \equiv 1$ for nonzero x , so $x^3 \equiv x^{-1}$. Everything is invertible $(\text{mod } 5)$ (except 0), so x^3 takes on all values $(\text{mod } 5)$. Very similar logic applies for 11, as $x^9 \equiv x^{-1} (\text{mod } 11)$ for all nonzero x , which means that 9th powers take on all values $(\text{mod } 11)$ (and therefore implies that the cubes take on all values $(\text{mod } 11)$). For 7, note that $x^4 \equiv x^{-2} (\text{mod } 7)$ for all nonzero x . This means x^4 takes on the values of all squares (which are 0, 1, 2, 4). The cubes equivalent to 1, 0, $-1 (\text{mod } 7)$, so just a little bit of checking shows that all residues modulo 7 are covered.

Now, we just have to check $b = 4, 8, 9$. 4 is easy to check. By Euler's Theorem, $x^4 \equiv 1 (\text{mod } 8)$ for all odd residues $(\text{mod } 8)$, so $x^3 \equiv x^{-1}$ and the cubes cover all odd residues mod 8. Since the quartic residues are 0 and 1, we cover everything. Finally, we just need to check $b = 9$. For all invertible elements x , $x^4 \equiv x^{-2}$, so we cover all squares. The squares $(\text{mod } 9)$ are 0, 1, 4, 7. The cubes contain 0, 1, and -1 so we hit everything. Therefore, $(7, 13)$ is the solution with smallest b . Thus the answer is $7 \cdot 13 = \boxed{91}$.

7. This sequence, which starts off as 1, 1, 1, 2, 3, 5, 21, 34, ... contains many members of the Fibonacci sequence. However, it is not the Fibonacci sequence. If $\{F_i\}_{i=1}^{\infty}$ is the Fibonacci sequence with $F_0 = F_1 = 1$, then the g sequence can be written as 1, 1, F_1 , F_2 , F_3 , F_4 , F_7 , F_8 , ..., which suggests that $g_{2k} = F_{2^k-1}$ and $g_{2k+1} = F_{2^k}$ for all positive integers k .

We prove this by induction. Since the initial conditions (g_0 and g_1) of the sequence are the same as F_1 and F_2 respectively, the base case is complete. Therefore, assume that $g_{2n} = F_{2^n-1}$ and $g_{2n+1} = F_{2^n}$ and consider g_{2n+2} in addition to g_{2n+3} . The Fibonacci identities $F_{2k} = F_k^2 + F_{k-1}^2$ and $F_{2k-1} = 2F_k F_{k-1} - F_{k-2}^2$ solve the inductive step.

For sake of completeness, we prove these identities in tandem by strong induction. For $F_{2k} = F_k^2 + F_{k-1}^2$, we can check that the base case $k = 1$ is consistent with this identity ($F_2 = 2 = 1 + 1 = F_1^2 + F_0^2$). For $F_{2k-1} = 2F_k F_{k-1} - F_{k-2}^2$, the base case of $k = 1$ is verified by $F_1 = 1 = 2(1) - 1 = 2F_1 F_0 - F_0^2$, as desired. Therefore, we have completed the base case for induction. For the inductive hypothesis, assume that

$$F_{2k-2} = F_{k-1}^2 + F_{k-2}^2 \text{ and } F_{2k-3} = 2F_{k-1}F_{k-2} - F_{k-2}^2.$$



Note that

$$\begin{aligned} 2F_k F_{k-1} - F_{k-1}^2 &= 2(F_{k-1} + F_{k-2})F_{k-1} - F_{k-1}^2 = F_{k-1}^2 + 2F_{k-1}F_{k-2} \\ &= (F_{k-1}^2 + F_{k-2}^2) + (2F_{k-1}F_{k-2} - F_{k-2}^2). \end{aligned}$$

By this equation and the inductive hypothesis,

$$2F_k F_{k-1} - F_{k-1}^2 = (F_{k-1}^2 + F_{k-2}^2) + (2F_{k-1}F_{k-2} - F_{k-2}^2) = F_{2k-2} + F_{2k-3} = F_{2k-1},$$

as desired. Now, we need to finish the inductive step for the other equation:

$$\begin{aligned} F_k^2 + F_{k-1}^2 &= (F_{k-1} + F_{k-2})^2 + F_{k-1}^2 = 2F_{k-1}^2 + 2F_{k-1}F_{k-2} + F_{k-2}^2 \\ &= 2(F_{k-1}^2 + F_{k-2}^2) + 2F_{k-1}F_{k-2} - F_{k-2}^2. \end{aligned}$$

By the inductive hypothesis,

$$\begin{aligned} F_k^2 + F_{k-1}^2 &= 2(F_{k-1}^2 + F_{k-2}^2) + 2F_{k-1}F_{k-2} - F_{k-2}^2 = 2F_{2k-2} + F_{2k-3} \\ &= F_{2k-1} + F_{2k-2} = F_{2k}, \end{aligned}$$

as desired. Therefore, the induction is complete and both identities have been proven. (These identities may also be proven by Binet's formula or by matrix products.)

Now, consider the Fibonacci Sequence (mod 8) and (mod 27). The first terms of the Fibonacci Sequence (mod 8) are 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, ... (repeat of the beginning) so the period of the Fibonacci Sequence (mod 8) is 12. $2^k \equiv 2^{k+2} \pmod{12}$ for all $k \geq 2$, so $g_{2011} = F_{2^{1005}} \equiv F_8 \equiv 2 \pmod{8}$. To calculate the period (mod 27), write out terms of the sequence (1, 1, 2, 3, 5, 8, 13, 21, 7, 1, 8, 0, ...). Note that $F_{12} \equiv 0 \pmod{27}$. Therefore, for all indices i such that $13 \leq i \leq 24$, $F_i \equiv 8F_{i-12} \pmod{27}$ since this part of the sequence is generated by $F_{12} \equiv 0$ and $F_{13} \equiv 8 \pmod{27}$. Therefore, $F_{23} \equiv 64 \equiv 10 \pmod{27}$. By the same logic as before, $F_i \equiv 10F_{i-24}$ if $25 \leq i \leq 36$. Therefore, $F_{35} \equiv 80 \equiv -1 \pmod{27}$. Therefore, for all i such that $36 \leq i \leq 71$, $F_i \equiv -F_{i-36} \pmod{27}$, so $F_{71} \equiv 1 \pmod{27}$. Since $F_{72} \equiv 0 \pmod{27}$, the Fibonacci sequence repeats after 72 terms. There is no smaller period, as that period would have to divide 36 or 24, even though we know that $F_{35} \equiv -1$ and $F_{23} \equiv 10 \pmod{27}$.

Now, $2^{1002} = 8^{334} \equiv (-1)^{334} \equiv 1 \pmod{9}$, so $2^{1005} \equiv 8 \pmod{72}$. Therefore, $F_{2^{1005}} \equiv F_8 \pmod{27}$. Therefore, $g_{2011} \equiv F_8 \equiv 34 \pmod{216}$, so the answer is 34.

8. Note that

$$\sum_{i=1}^m i^3 = \left(\sum_{i=1}^m i \right)^2$$

for all positive integers m . Therefore,

$$\sum_{i=k+1}^n i^3 = \left(\frac{n(n+1)}{2} \right)^2 - \left(\frac{k(k+1)}{2} \right)^2.$$



The given equation therefore is the same as

$$\left(\frac{n(n+1)}{2}\right)^2 = (96^2(3)) \left(\frac{k(k+1)}{2}\right)^2 + 48^2.$$

Since 48^2 divides the right side of this equation, $48 \mid \frac{n(n+1)}{2}$, so we can write this equation as

$$\left(\frac{n(n+1)}{96}\right)^2 - 3(k(k+1))^2 = 1.$$

This is Pell's equation. Therefore, $\frac{n(n+1)}{96}$ can be written as $\frac{(2+\sqrt{3})^m + (2-\sqrt{3})^m}{2}$ for any positive integer m . Furthermore, $k(k+1) = \frac{(2+\sqrt{3})^m - (2-\sqrt{3})^m}{2\sqrt{3}}$. Since $k \equiv 0, 3 \pmod{4}$, $4 \mid k(k+1)$ and $8 \mid \frac{(2+\sqrt{3})^m - (2-\sqrt{3})^m}{\sqrt{3}}$. Note that

$$\frac{(2+\sqrt{3})^m - (2-\sqrt{3})^m}{\sqrt{3}} \equiv 2(3)^{\lfloor \frac{m}{2} \rfloor} \pmod{8}$$

for odd m (since all terms besides the last term in the expansion are divisible by 8). This contradicts the fact that $k(k+1)$ is a multiple of 4, so m is even.

For even m , let $m = 2p$ for some integer p . Then

$$\frac{n(n+1)}{96} = \frac{(2+\sqrt{3})^m + (2-\sqrt{3})^m}{2}$$

is equivalent to

$$4n^2 + 4n = 192((2+\sqrt{3})^{2p} + (2-\sqrt{3})^{2p}) = 64 \left(\sqrt{3}((2+\sqrt{3})^p - (2-\sqrt{3})^p) \right)^2 + 384.$$

Note that $\sqrt{3}((2+\sqrt{3})^p - (2-\sqrt{3})^p)$ is a positive integer, so WLOG call it x . Then $(2n+1)^2 = 64x^2 + 385$, so from difference of squares $(2n+1-8x)(2n+1+8x) = 5 \cdot 7 \cdot 11$. The only positive integer solutions to this equation are $(n, x) = (96, 24)$ and $(15, 3)$. $3 \neq \sqrt{3}((2+\sqrt{3})^p - (2-\sqrt{3})^p)$ for any p , but $p = 2$ yields the $x = 24$ case. Therefore, the only possible solution occurs when $n = 96$. Plugging this into the original equation shows that $k = 7$ so the answer is $96+7 = \boxed{103}$.