# PUMaC 2011

## Number Theory A

1. [**3**] The only prime factors of an integer $n$ are 2 and 3. If the sum of the divisors of $n$ (including itself) is 1815, find $n$.

2. [**3**] What is the largest positive integer $n < 1000$ for which there is a positive integer $m$ satisfying
$$\operatorname{lcm}(m,n) = 3m \times \gcd(m,n)?$$

3. [**4**] What is the sum of all primes $p$ such that $7^p - 6^p + 2$ is divisible by 43?

4. [**4**] For how many ordered triplets of three positive integers is it true that their product is four more than twice their sum?

5. [**5**] Let $d(n)$ denote the number of divisors of $n$ (including itself). You are given that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$
Find $p(6)$, where $p(x)$ is the unique polynomial with rational coefficients satisfying
$$p(\pi) = \sum_{n=1}^{\infty} \frac{d(n)}{n^2}.$$

6. [**6**] Let $a$ and $b$ be positive integers such that $a + bz = x^3 + y^4$ has no solutions for any integers $x, y, z$, with $b$ as small as possible, and $a$ as small as possible for the minimum $b$. Find $ab$.

7. [**7**] Let $\{g_i\}_{i=0}^{\infty}$ be a sequence of positive integers such that $g_0 = g_1 = 1$ and the following recursions hold for every positive integer $n$:
$$g_{2n+1} = g_{2n-1}^2 + g_{2n-2}^2$$
$$g_{2n} = 2g_{2n-1}g_{2n-2} - g_{2n-2}^2$$
Compute the remainder when $g_{2011}$ is divided by 216.

8. [**8**] Calculate the sum of the coordinates of all pairs of positive integers $(n, k)$ such that
$k \equiv 0, 3 \pmod 4$, $n > k$, and $\displaystyle\sum_{i=k+1}^{n} i^3 = (96^2 \cdot 3 - 1)\left(\sum_{i=1}^{k} i\right)^2 + 48^2$.