PUMaC 2011

Geometry A Solutions

www.hanlin.io 仅内部使用,严禁商用

1. Note that the solid formed is a generalized cylinder. It is clear from the diagram that the area of the base of this cylinder (i.e., a vertical cross-section of the log) is composed of two semicircles of radius 3 and a part of an annulus. In the right triangle in the diagram, the hypotenuse is 4 and the vertical leg is 2. Thus, it is a 30-60-90 triangle, so the central angle in the annulus is 120°. Since the annular region has inner radius 1 and outer radius 7, the total area is $2(\frac{1}{2}\pi 3^2) + \frac{1}{3}\pi(7^2 - 1^2) = 25\pi$. Hence the volume of the cylinder is $10 \cdot 25\pi = 250\pi$, so the answer is $\boxed{250}$.

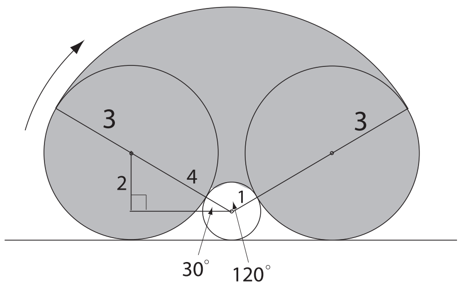

Figure 1: Problem 1 diagram

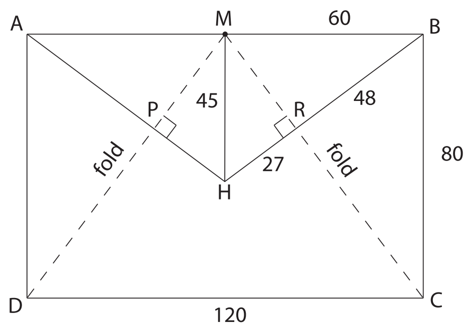

Figure 2: Problem 2 diagram

2. Pick P on DM and R on CM so the AP is perpendicular to DM and BR is perpendicular to CM. Because of the way the paper is being folded, the projection of A onto the plane of the paper is always along line AP, and the projection of B along line BR. Thus, the two lines will intersect in exactly the point H. Since $\triangle HMB \sim \triangle MBC$, we have $HM/MB = MB/BC$, so $HM = (MB/BC) \cdot MB = (60/80) \cdot 60 = (3/4) \cdot 60 = \boxed{45}$.

Institute
www.hanlin.io
仅内部使用,严禁商用

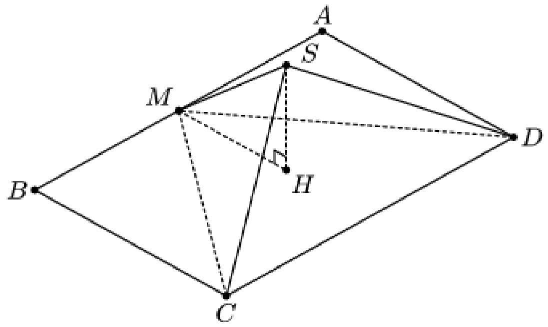

Figure 3: Problem 2 Diagram

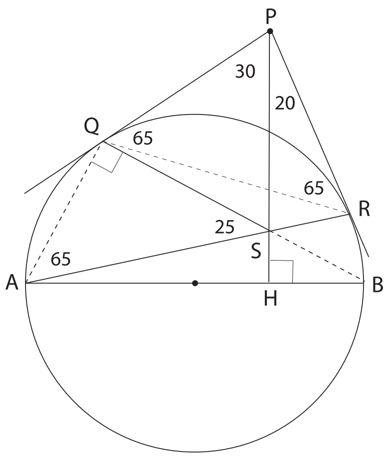

Figure 4: Problem 3 diagram.

3. We claim that $PS = PR$. To see this, let $x = \angle RAB$. Then, calculating arc measures gives that $\angle PRS = 90° - x$. Also, from right triangle ASH, we have that $\angle PSR = \angle ASH = 90° - x$. Thus, $PR = PS$. It also follows from the angles in triangle PSR that $x = 10°$. Now, since PR and PQ are tangents to the circle, we have $PQ = PS = PR$. Thus, there is a circle centered at P passing through Q, S, and R. Then we can obtain that

$$\angle QSA = \angle SQR + \angle QRS = \frac{1}{2}\angle SPR + \frac{1}{2}\angle SPQ = 15° + 10° = \boxed{25°}.$$

It is interesting to note that the points Q, S, B are collinear because $\angle RQB = 10° = \angle RAB$. Hence $\angle AQB = 90°$, from which another solution can be found.

4. **First solution**: We claim that BP is perpendicular to AI. Let M be the intersection of lines BP and AI. We have that $\angle IBM = \angle IBP = \angle ICP$. Also, $\angle BIM = \angle ABI + \angle IAB$, so

$$\angle IBM + \angle BIM = \angle ICP + \angle ABI + \angle IAB = \frac{1}{2}(\angle A + \angle B + \angle C) = 90°.$$

www.hanlin.io 仅内部使用,严禁商用

Thus, BP is indeed perpendicular to AI. Thus, triangles ABM and APM are congruent, so $AP = AB = 15$, so $PC = AC - AP = 21 - 15 = \boxed{6}$.

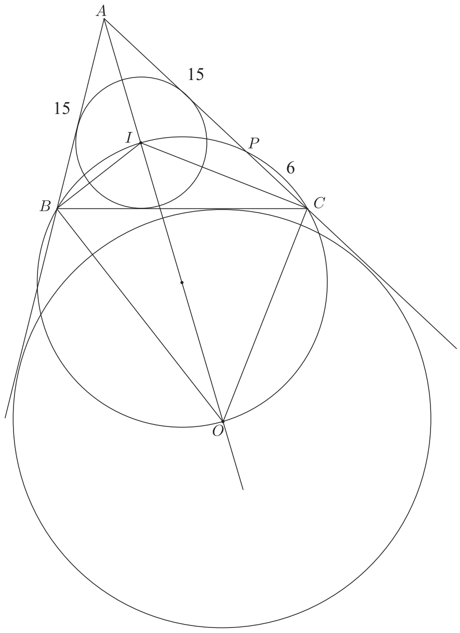

Figure 5: Problem 4 diagram.

Second solution: Let ω be the circumcircle of triangle IBC. We claim that ω is the circle with diameter IO, where O is the excenter of ABC corresponding to A. Draw the external angle bisectors at vertices B and C. These two lines intersect at O. Moreover, since IC is perpendicular to CO and IB is perpendicular to BO, so quadrilateral $IBOC$ is cyclic, and its circumcircle is precisely ω. Thus, since I and O lie on the angle bisector of $\angle BAC$, the circle ω lies symmetric to $\angle BAC$. Thus, if $AB = 15$, then $AP = 15$ as well, from which is follows that $CP = AC - AP = 21 - 15 = \boxed{6}$.

5. Without loss of generality, suppose A lies to the left of B. Let D' be the point such that $DAD'B$ is a parallelogram. No matter what the positions of A and B are, we have that $BD = 15/\sin(60°) = 10\sqrt{3}$, $AC = 15/\sin(30°) = 30$, and $\angle CAD' = \angle CAB + \angle BAD' = \angle CAB + \angle DBA = 90°$. Thus, CD' is always $20\sqrt{3}$ as A and B vary. Note that $AD + BC = BD' + BC$. By the triangle inequality, this length is no less that $CD' = 20\sqrt{3}$, and equality can be achieved by fixing A and moving B to the intersection of CD' with ℓ_1. Thus, $20\sqrt{3}$ is the minimum length, so the answer is $20 + 3 = \boxed{23}$.

www.hanlin.io 仅内部使用,严禁商用 Institute

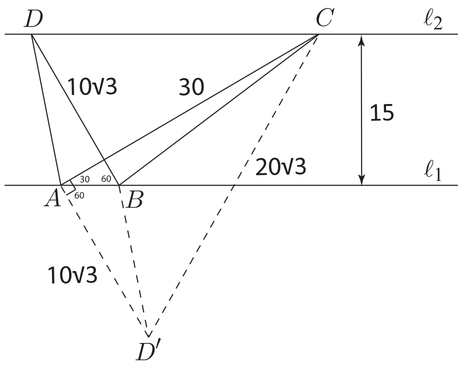

Figure 6: Problem 5 diagram.

6. We claim that the length of arc MN is constant as P varies. We can see this by noting that $\widehat{MLB} - \widehat{AN} = \frac{1}{2}\angle APB$, which is constant, and that $\widehat{MLB} + \widehat{MA}$ is constant. Subtracting these two constant quantities, we get that $\widehat{MN} = \widehat{MA} + \widehat{AN}$ is constant. Since OS is the distance from O to the midpoint of a chord of constant length, OS is constant as well. Thus, the locus of all points S is a part of a circle centered at O. It follows that the minimum distance from this locus to point A is the difference between the radii of ω_1 and of the locus of S. Now, to find the radius of the locus of S, consider the location of S when P is at the midpoint C of major arc AB. Since ω_2 passes through O, we have that CA and CB are tangent to ω_1. Thus, the segment MN becomes AB, and S coincides with T, the midpoint of AB. By the similarity of triangles TAO and ACO, we have that $OT/OA = OA/OC$, so $OT = OA^2/OC = 6^2/10 = 18/5$. Thus, the radius of the locus of S is $18/5$, and the difference between the two radii is $6 - 18/5 = 12/5$, so the answer is $12 + 5 = \boxed{17}$.

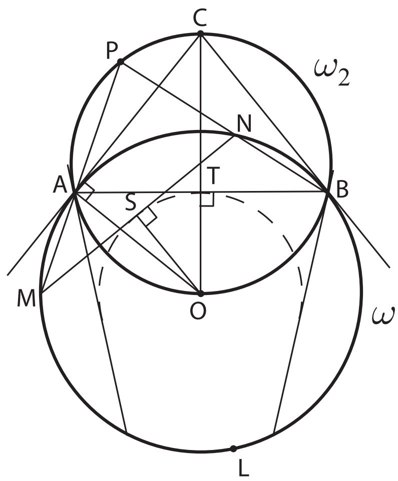

Figure 7: Problem 6 diagram.

www.hanlin.io 仅内部使用,严禁商用

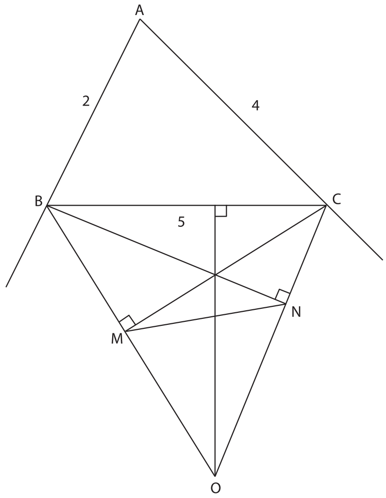

Figure 8: Problem 7 diagram.

7. **First Solution**: Extend BM and CN to meet at the excenter O. Let $[P_1P_2...P_n]$ denote the area of polygon $P_1P_2...P_n$. Since quadrilateral $BMNC$ is cyclic, we have that triangle OMN is similar to triangle OCB. Thus, we have that $[OMN]/[OCB] = (ON/OB)^2 = \cos^2(\angle O)$. We have

$$\begin{aligned}\angle O &= 180^\circ - (\angle CBO + \angle BCO)\\ &= 180^\circ - [(180^\circ - \angle ABC)/2 + (180^\circ - \angle ACB)/2]\\ &= (\angle ABC + \angle ACB)/2\\ &= (180^\circ - \angle A)/2\\ &= 90^\circ - \angle A/2.\end{aligned}$$

Thus, $\cos^2(\angle O) = \cos^2(90^\circ - \angle A/2) = \sin^2(\angle A/2) = \frac{1}{2}(1 - \cos\angle A)$. By the cosine theorem for triangle ABC, we have $\cos\angle A = (AB^2 + AC^2 - BC^2)/(2 \cdot AB \cdot AC) = (4 + 16 - 25)/16 = -5/16$. Thus, $[OMN]/[OCB] = \cos^2\angle O = \frac{1}{2}(1 - \cos\angle A) = \frac{1}{2}(1 + \frac{5}{16}) = \frac{21}{32}$. It follows that $[BMNC]/[OCB] = 1 - [OMN]/[OCB] = 1 - \frac{21}{32} = \frac{11}{32}$. Thus,

$$\frac{[BMNC]}{[ABC]} = \frac{[BMNC]}{[OCB]} \cdot \frac{[OCB]}{[ABC]} = \frac{11}{32} \cdot \frac{\frac{1}{2} \cdot CB \cdot r_A}{[ABC]},$$

where r_A is the altitude from O of triangle OBC, which is the exradius corresponding to A. This exradius is $r_A = [ABC]/(s - a)$, where s is the semiperimeter of triangle ABC, and $a = BC$. Thus, $r_A/[ABC] = 1/(s - a) = 1/[(2 + 4 - 5)/2] = 2$. Finally, we find that

$$\frac{[BMNC]}{[ABC]} = \frac{11}{32} \cdot \frac{\frac{1}{2} \cdot CB \cdot r_A}{[ABC]} = \frac{11}{64} \cdot 5 \cdot \frac{r_A}{[ABC]} = \frac{11}{64} \cdot 5 \cdot 2 = \frac{55}{32}.$$

Thus, the answer is $55 + 32 = \boxed{87}$.

Institute www.hanlin.io 仅内部使用,严禁商用

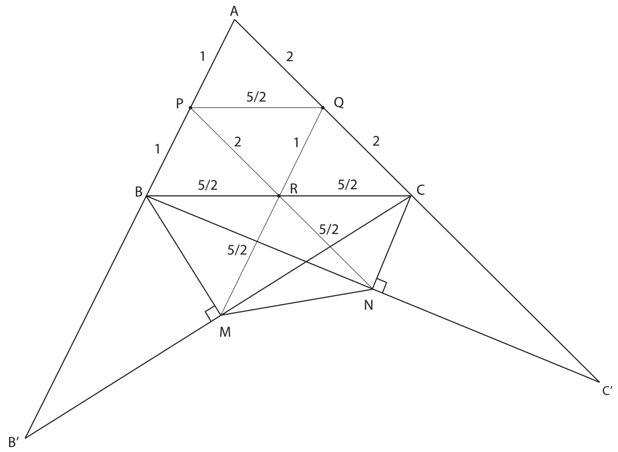

Figure 9: Problem 7 diagram.

Second Solution: Extend CM to meet AB at B' and BN to meet AC at C'. Then, $B'M = MC$ and $C'N = NB$. Let P, Q, R be the midpoints of AB, AC, BC. Then, we see that M, R, Q all lie on the midline of triangle $AB'C$. Similarly, P, R, N all lie on the midline of triangle ABC'. Observe that $[PBR] = [PRQ] = [RQC] = 1/4[ABC]$. We have that

$$\begin{aligned}
\frac{[BCNM]}{[ABC]} &= \frac{[BMR]}{[ABC]} + \frac{[MRN]}{[ABC]} + \frac{[RNC]}{[ABC]} \\
&= \frac{1}{4}\frac{[BMR]}{[RQC]} + \frac{1}{4}\frac{[MRN]}{[PQR]} + \frac{1}{4}\frac{[RNC]}{[PRB]} \\
&= \frac{1}{4}\left(\frac{RB \cdot RM}{RQ \cdot RC} + \frac{RN \cdot RM}{RP \cdot RQ} + \frac{RN \cdot RC}{RP \cdot RB}\right) \\
&= \frac{1}{4}\left(\frac{BC^2}{AB \cdot BC} + \frac{BC^2}{AB \cdot AC} + \frac{BC^2}{AC \cdot BC}\right) \\
&= \frac{1}{4}BC^2 \cdot \frac{AC + BC + AB}{AB \cdot BC \cdot AC} \\
&= \frac{1}{4}BC \cdot \frac{AC + BC + AB}{AB \cdot AC} \\
&= \frac{1}{4} \cdot 5 \cdot \frac{11}{2 \cdot 4} = \frac{55}{32},
\end{aligned}$$

so the answer is $55 + 32 = \boxed{87}$.

8. Calculating side BC using the Theorem of Cosines, we get that $BC = \sqrt{7}$. Then, calculating $\angle BMC$ using the Theorem of Cosines in triangle BMC, we get that $\angle BMC = 120°$. Now, we reflect triangle BMC over line BC, and let D be the reflection of M. Note that quadrilateral $BDCM$ is a kite with the circle inscribed in it. Now, consider quadrilateral $ABDC$. It has opposite angles adding to $180°$, so it is an inscribed quadrilateral. Since $AB = DC = 2$, it

www.hanlin.io 仅内部使用,严禁商用

is an isosceles trapezoid. Moreover, $AB + DC = 4 = BD + AC$ implies that $ABDC$ is a circumscribed quadrilateral. Note that because the perpendicular bisector of AC is the line of symmetry of $ABDC$, it passes through the incenter of $ABDC$. Also, the angle bisector of $\angle BAC$ passes through the incenter of $ABDC$. Thus, the point P in the problem is actually the incenter of $ABDC$. Note that points O and P both lie on the angle bisector of $\angle BDC$, so D, O, P are collinear. Moreover, by symmetry, $DO = OM$, so $DO/DP = MO/(MO + OP)$. Thus, $(MO + OP)/MO = DP/DO$, so $OP/MO = DP/DO - 1$. By the homothety centered at point D, DP/DO is the same as the ratio of the radii of the two circles. To find the radius of the larger circle, we consider the 30°-60°-90° right triangle with hypotenuse AP. From this, the larger radius can immediately be seen to be $\sqrt{3}/2$. To find the radius of the smaller circle, consider the area S of triangle BMC. If r is the radius of the smaller circle, then $S = \frac{1}{2}(1 \cdot r + 2 \cdot r) = \frac{3}{2}r$. On the other hand, the area of triangle BMC is $S = \frac{1}{2}BM \cdot MC \sin(\angle BMC) = \frac{1}{2} \cdot 1 \cdot 2 \cdot \sin(120°) = \sqrt{3}/2$. Equating the two expressions for S, we get that $r = \sqrt{3}/3$. Thus, $DP/DO = (\sqrt{3}/2)/(\sqrt{3}/3) = 3/2$. Thus, $OP/MO = DP/DO - 1 = 3/2 - 1 = 1/2$. Thus, our answer is $1 + 2 = \boxed{3}$.

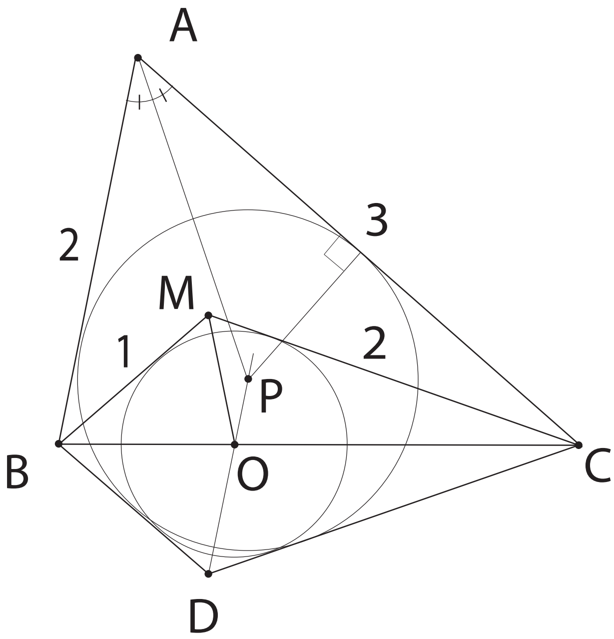

Figure 10: Problem 8 diagram.