PUMaC 2011

Geometry A

1. [**3**] Two logs of length 10 are laying on the ground touching each other. Their radii are 3 and 1, and the smaller log is fastened to the ground. The bigger log rolls over the smaller log without slipping, and stops as soon as it touches the ground again. The volume of the set of points swept out by the larger log as it rolls over the smaller one can be expressed as $n\pi$, where n is an integer. Find n.

2. [**3**] A rectangular piece of paper has corners labeled A, B, C, and D, with $BC = 80$ and $CD = 120$. Let M be the midpoint of side AB. The corner labeled A is folded along line MD and the corner labeled B is folded along line MC until the segments AM and MB coincide. Let S denote the point in space where A and B meet. If H is the foot of the perpendicular from S to the original plane of the paper, find HM.

3. [**4**] Let PQ and PR be tangents to a circle ω with diameter AB so that A, Q, R, B lie on ω in that order. Let H be the projection of P onto AB and let AR and PH intersect at S. If $\angle QPH = 30^\circ$ and $\angle HPR = 20^\circ$, find $\angle ASQ$ in degrees.

4. [**4**] Let ABC be a triangle with $AB = 15, BC = 17$, $CA = 21$, and incenter I. If the circumcircle of triangle IBC intersects side AC again at P, find CP.

5. [**5**] Let ℓ_1 and ℓ_2 be two parallel lines, a distance of 15 apart. Points A and B lie on ℓ_1 while points C and D lie on ℓ_2 such that $\angle BAC = 30^\circ$ and $\angle ABD = 60^\circ$. The minimum value of $AD + BC$ is $a\sqrt{b}$, where a and b are integers and b is squarefree. Find $a + b$.

6. [**6**] Let ω_1 be a circle of radius 6, and let ω_2 be a circle of radius 5 that passes through the center O of ω_1. Let A and B be the points of intersection of the two circles, and let P be a point on major arc AB of ω_2. Let M and N be the second intersections of PA and PB with ω_1, respectively. Let S be the midpoint of MN. As P ranges over major arc AB of ω_2, the minimum length of segment SA is a/b, where a and b are positive integers and $\gcd(a, b) = 1$. Find $a + b$.

7. [**7**] Let ABC be a triangle with $AB = 2, BC = 5, AC = 4$. Let M be the projection of C onto the external angle bisector at vertex B. Similarly, let N be the projection of B onto the external angle bisector at vertex C. If the ratio of the area of quadrilateral $BCNM$ to the area of triangle ABC is a/b, where a and b are positive integers and $\gcd(a, b) = 1$, find $a + b$.

8. [**8**] Let ABC be a triangle with $\angle BAC = 60^\circ, BA = 2$, and $CA = 3$. A point M is located inside ABC such that $MB = 1$ and $MC = 2$. A semicircle tangent to MB and MC has its center O on BC. Let P be the intersection of the angle bisector of $\angle BAC$ and the perpendicular bisector of AC. If the ratio OP/MO is a/b, where a and b are positive integers and $\gcd(a, b) = 1$, find $a + b$.