PUMaC 2010

Combinatorics B

1. The Princeton University Band plays a setlist of 8 distinct songs, 3 of which are tiring to play. If the Band can't play any two tiring songs in a row, how many ways can the band play its 8 songs?

2. PUMaCDonalds, a newly-opened fast food restaurant, has 5 menu items. If the first 4 customers each choose one menu item at random, the probability that the 4th customer orders a previously unordered item is m/n, where m and n are relatively prime positive integers. Find $m + n$.

3. Let $\underline{xyz}$ represent the three-digit number with hundreds digit x, tens digit y, and units digit z, and similarly let $\underline{yz}$ represent the two-digit number with tens digit y and units digit z. How many three-digit numbers $\underline{abc}$, none of whose digits are 0, are there such that $\underline{ab} > \underline{bc} > \underline{ca}$?

4. Sterling draws 6 circles on the plane, which divide the plane into regions (including the unbounded region). What is the maximum number of resulting regions?

5. $3n$ people take part in a chess tournament: n girls and $2n$ boys. Each participant plays with each of the others exactly once. There were no ties and the number of games won by the girls is $\dfrac{7}{5}$ the number of games won by the boys. How many people took part in the tournament?

6. A regular pentagon is drawn in the plane, along with all its diagonals. All its sides and diagonals are extended infinitely in both directions, dividing the plane into regions, some of which are unbounded. An ant starts in the center of the pentagon, and every second, the ant randomly chooses one of the edges of the region it's in, with an equal probability of choosing each edge, and crosses that edge into another region. If the ant enters an unbounded region, it explodes. After first leaving the central region of the pentagon, let x be the expected number of times the ant re-enters the central region before it explodes. Find the closest integer to $100x$.

7. We say that a rook is "attacking" another rook on a chessboard if the two rooks are in the same row or column of the chessboard and there is no piece directly between them. Let n be the maximum number of rooks that can be placed on a 6×6 chessboard such that each rook is attacking at most one other. How many ways can n rooks be placed on a 6×6 chessboard such that each rook is attacking at most one other?

8. Matt is asked to write the numbers from 1 to 10 in order, but he forgets how to count. He writes a permutation of the numbers $\{1, 2, 3 \ldots, 10\}$ across his paper such that:

   (a) The leftmost number is 1.

   (b) The rightmost number is 10.

   (c) Exactly one number (not including 1 or 10) is less than both the number to its immediate left and the number to its immediate right.

   How many such permutations are there?