

# PUMaC 2010



## Algebra B Solutions

1. Let the operation  $\star$  be defined by  $x \star y = y^x - x * y$ . Calculate  $(3 \star 4) - (4 \star 3)$ .

Solution: -17. We have  $(3 \star 4) - (4 \star 3) = (64 - 12) - (81 - 12) = -17$ .

2. Let  $p(x) = x^2 + x + 1$ . Find the fourth smallest prime  $q$  such that  $p(x)$  has a root mod  $q$ .

Solution: 19. One can check that there are roots mod 3, 7, 13, and 19, and no others for smaller primes.

3. Write  $\frac{1}{\sqrt[5]{2}-1} = a + b\sqrt[5]{2} + c\sqrt[5]{4} + d\sqrt[5]{8} + e\sqrt[5]{16}$ , with  $a, b, c, d$ , and  $e$  integral. Find  $a^2 + b^2 + c^2 + d^2 + e^2$ .

Solution: 5. By multiplying both sides by  $\sqrt[5]{2} - 1$  and noting that the numbers 1,  $\sqrt[5]{2} = 2^{1/5}$ ,  $\sqrt[5]{4} = 2^{2/5}$ ,  $\sqrt[5]{8} = 2^{3/5}$ , and  $\sqrt[5]{16} = 2^{4/5}$  are all linearly independent over  $\mathbb{Q}$ , we can set up five equations for five unknowns, whose solution is  $a = b = c = d = e = 1$ .

4. Find the nearest integer to the sum of all  $x$  where  $4^x = x^4$ .

Solution: 5. We immediately see two solutions, 2 and 4, and that there can be no more positive roots. There must be a negative root, however: let  $f(x) = 4^x$  and  $g(x) = x^4$ ; then  $g(0) = 0$  and  $f(0) = 1$ , but  $g$  goes off to infinity as  $x \rightarrow -\infty$  and  $f$  goes to 0 as  $x \rightarrow \infty$ . Plugging in  $x = -1$ , we have  $f(-1) = 1/4$  and  $g(-1) = 1$ ; plugging in  $x = -1/2$  we have  $f(-1/2) = 1/2$  and  $g(-1/2) = 1/16$ . Therefore the root is between  $-1/2$  and  $-1$ , and the nearest integer to the sum of the roots must be 5.

5. Let  $x$  be a real root of the polynomial  $p(x) = x^3 - 3x + 3$ . Find  $x^9 + 81x^2$ .

Solution: This problem has been redacted. There is no integral solution to this problem.

6. Define  $f(x) = x + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \dots}}}}$ . Find the smallest integral  $x$  such that  $f(x) \geq 50\sqrt{x}$ .

Solution: 2400. Noting that  $(f(x) - x)^2 = f(x)$ , we can solve the quadratic equation for  $f(x)$  to get that

$$f(x) = x + \frac{1}{2} \pm \sqrt{x + \frac{1}{4}}.$$

We clearly have to take the positive root (we can notice, for example, that  $f(1) > 1$ ). The problem therefore reduces to finding the smallest integral  $x$  such that

$$x + \frac{1}{2} + \sqrt{x + \frac{1}{4}} \geq 50\sqrt{x}.$$

It is simple to note that  $x$  has to be fairly large for this to be satisfied (after trying the trivial  $x = 1$ ). For large  $x$ ,  $\sqrt{x + \frac{1}{4}}$  is very, very close to  $\sqrt{x}$ , so we can rewrite this as

$$x + \frac{1}{2} \geq 49\sqrt{x}.$$



The above is again rewritten as

$$x^2 - 2400x + \frac{1}{4} \geq 0.$$

The smallest integer  $x$  satisfying the above is obviously 2400, and since the margin of error here is  $\frac{1}{4}$ , our previous approximation is justified.

Of note is that this problem is cooked: the value  $x = 0$  is also a valid solution. We accepted either solution.

7. Let  $f$  be a function such that  $f(x) + f(x+1) = 2^x$  and  $f(0) = 2010$ . Find the last two digits of  $f(2010)$ .

Solution: 51. We have the sequence of equations

$$\begin{aligned} f(x) + f(x+1) &= 2^x \\ f(x+1) + f(x+2) &= 2^{x+1} = 2 \cdot 2^x \\ &\dots \\ f(x+n-1) + f(x+n) &= 2^{x+n-1} = 2^{n-1} \cdot 2^x. \end{aligned}$$

Adding and subtracting alternate lines, we get a telescoping sum:

$$f(x) + (-1)^{n+1} f(x+n) = 2^x \sum_{k=0}^{n-1} 2^k (-1)^k = 2^x \sum_{k=0}^{n-1} (-2)^k = 2^x \left( \frac{1 - (-2)^n}{3} \right).$$

Plug in  $x = 0$  and  $n = 2010$ , so

$$f(2010) = \frac{2^{2010} - 1}{3} + 2010.$$

The last two digits of  $2^{2010}$  are 24 (using Euler's theorem with  $n = 25$ , we have  $2^{20} = 1 \pmod{25}$ , so  $2^{2000} = 1 \pmod{25}$ , so  $2^{2010} = 2^{10} \pmod{25}$ , so  $2^{2010} = 24 \pmod{100}$ ). Therefore the expression  $(2^{2010} - 1)/3$  has last digits 41, so overall the last two digits are 51.

8. The expression  $\sin 2^\circ \sin 4^\circ \sin 6^\circ \cdots \sin 90^\circ$  is equal to  $p\sqrt{5}/2^{50}$ , where  $p$  is an integer. Find  $p$ .

Solution: 192. Let  $\omega$  be the root of unity  $e^{2\pi i/90}$ , so we have

$$\prod_{n=1}^{45} \sin(2n^\circ) = \sum_{n=1}^{45} \frac{\omega^n - 1}{2i\omega^{n/2}}.$$

By the symmetry of the sine (and the fact that  $\sin(90^\circ) = 1$ ),

$$\prod_{n=1}^{45} \sin(2n^\circ) = \prod_{n=46}^{89} \sin(2n^\circ),$$

so

$$\left| \prod_{n=1}^{45} \sin(2n^\circ) \right|^2 = \sum_{n=1}^{89} \frac{|\omega^n - 1|}{2} = \frac{90}{2^{89}},$$



where we've used the usual geometric series sum for roots of unity. The product is clearly positive and real, so it is equal to

$$\frac{\sqrt{45}}{2^{44}} = \frac{3\sqrt{5}}{2^{44}},$$

implying that  $p = 3 \cdot 2^6 = 192$ .