PUMaC 2010

Number Theory A Solutions

1. Find the smallest positive integer n such that $n^4 + (n+1)^4$ is composite.

   Solution: One can check that 17, 97, 337, and 881 are prime, and also that $17 \cdot 113 = 1921$, so the answer is 5.

2. Find the largest positive integer n such that $\sigma(n) = 28$, where $\sigma(n)$ is the sum of the divisors of n, including n.

   Solution: The answer is 12. It isn't hard to see that that is, in fact, the only value where that equation holds.

3. Find the sum of the first 5 positive integers n such that $n^2 - 1$ is the product of 3 distinct primes.

   Solution: The five numbers are 14, 16, 20, 22, and 32, for an answer of 104.

4. Find the largest positive integer n such that $n\varphi(n)$ is a perfect square. ($\varphi(n)$ is the number of integers k, $1 \le k \le n$ that are relatively prime to n)

   Solution: If $n > 1$, then the highest prime that divides n can be shown to divide $n\varphi(n)$ to an odd power, and so $n\varphi(n)$ cannot be a perfect square. It is easy to see that 1 is a perfect square.

5. Given that x, y are positive integers with x as small as possible, and y minimized with that constraint, and $x(x+1)|y(y+1)$, but neither x nor $x+1$ divides either y or $y+1$, find x^2+y^2.

   Solution: $x = 14$, $y = 20$. These are the first two times that n and $n+1$ are not powers of primes. The answer is then easily seen to be 596.

6. Find the numerator of

$$\frac{1010\overbrace{11\ldots11}^{2011 \text{ ones}}0101}{1100\underbrace{11\ldots11}_{2011 \text{ ones}}0011}$$

   when reduced to lowest terms.

   Solution: Note that $1+x^2+x^4+x^5+\ldots+x^{2n+1}+x^{2n+2}+x^{2n+4}+x^{2n+6} = (1-x+x^2-x^3+x^4)(1+x+\ldots+x^{2n+1}+x^{2n+2})$, as well as $1+x+x^4+x^5+\ldots+x^{2n+1}+x^{2n+2}+x^{2n+5}+x^{2n+6} = (1-x^2+x^4)(1+x+\ldots+x^{2n+1}+x^{2n+2})$. The easiest way to see this is to either multiply numerator and denominator by $x-1$, or to just numerically plug in small odd values of 2011 in the original equation. Plugging in $n = 1006$ and $x = 10$ gives the fraction as $\frac{9091}{9901}$.

7. Let $I = \{0, 1, 2, \ldots 2008, 2009\}$, and let $S = \{f : I \to I | f(a) \equiv g(a) (\text{mod } 2010) \forall a\}$, where $g(a)$ ranges over all polynomials with integer coefficients. The number of elements in S can be written as $p_1p_2\cdots p_k$, where the p_is are (not necessarily distinct) primes. Find $p_1+p_2+\cdots+p_n$.

   Solution: Any polynomial is determined, by the Chinese remainder theorem, by its restriction to the integers mod 2, 3, 5, and 67. There are thus at most $2^2 \cdot 3^3 \cdot 5^5 \cdot 67^{67}$ such functions. However, if you have a polynomial in those various moduli, then there is a polynomial that restricts to those polynomials, so there are at least $2^2 \cdot 3^3 \cdot 5^5 \cdot 67^{67}$ such functions and thus exactly $2^2 \cdot 3^3 \cdot 5^5 \cdot 67^{67}$ such functions. The answer is then 4527.

8. An awesome pythagorean triple is a pythagorean triple of the form $a^2 + (a+1)^2 = b^2$, a and b positive integers. Given that a, $a+1$, and b form the third positive awesome pythagorean triple, find a.

Solution: The equation is $2a^2 + 2a + 1 = b^2$, which can be rearranged to $(2a+1)^2 - 2b^2 = -1$. The solutions to that are given by the coefficients of $(1+\sqrt{2})(3+2\sqrt{2})^n$, and one gets a values 0 (trivial and ignored), 3, 20, and 119, so 119 is the answer.