PUMaC 2010

Algebra A Solutions

1. Find the sum of the coefficients of the polynomial $(63x-61)^4$.

   Solution: 16. The sum of the coefficients of $f(x)=(63x-61)^4$ is $f(1)=(63-61)^4=16$.

2. Calculate $\sum_{n=1}^{\infty}\left(\lfloor\sqrt[n]{2010}\rfloor-1\right)$.

   Solution: 2077. Just calculate it; note that for $n>10$ we have $2^{10}>2010$ so all those terms are zero. We get $2009+43+11+5+3+2+1+1+1+1=2077$.

3. Find the nearest integer to the sum of all x where $4^x=x^4$.

   Solution: 5. We immediately see two solutions, 2 and 4, and that there can be no more positive roots. There must be a negative root, however: let $f(x)=4^x$ and $g(x)=x^4$; then $g(0)=0$ and $f(0)=1$, but g goes off to infinity as $x\to-\infty$ and f goes to 0 as $x\to\infty$. Plugging in $x=-1$, we have $f(-1)=1/4$ and $g(x)=1$; plugging in $x=-1/2$ we have $f(-1/2)=1/2$ and $g(x)=1/16$. Therefore the root is between $-1/2$ and -1, and the nearest integer to the sum of the roots must be 5.

4. Define $f(x)=x+\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}$. Find the smallest integral x such that $f(x)\geq 50\sqrt{x}$.

   Solution: 2400. Noting that $(f(x)-x)^2=f(x)$, we can solve the quadratic equation for $f(x)$ to get that

   $$f(x)=x+\frac{1}{2}\pm\sqrt{x+\frac{1}{4}}.$$

   We clearly have to take the positive root (we can notice, for example, that $f(1)>1$). The problem therefore reduces to finding the smallest integral x such that

   $$x+\frac{1}{2}+\sqrt{x+\frac{1}{4}}\geq 50\sqrt{x}.$$

   It is simple to note that x has to be fairly large for this to be satisfied (after trying the trivial $x=1$). For large x, $\sqrt{x+\frac{1}{4}}$ is very, very close to $\sqrt{x}$, so we can rewrite this as

   $$x+\frac{1}{2}\geq 49\sqrt{x}.$$

   The above is again rewritten as

   $$x^2-2400x+\frac{1}{4}\geq 0.$$

   The smallest integer x satisfying the above is obviously 2400, and since the margin of error here is $\frac{1}{4}$, our previous approximation is justified.

5. Let $f(x) = 3x^3 - 5x^2 + 2x - 6$. If the roots of f are given by α, β, and γ, find

$$\left(\frac{1}{\alpha-2}\right)^2 + \left(\frac{1}{\beta-2}\right)^2 + \left(\frac{1}{\gamma-2}\right)^2.$$

Solution: 68. A polynomial with roots $\alpha-2$, $\beta-2$, and $\gamma-2$ is given by

$$g(x) = f(x+2) = 3x^3 + 13x^2 + 18x + 2.$$

A polynomial with roots $1/(\alpha-2)$, $1/(\beta-2)$, and $1/(\gamma-2)$ is given by

$$h(x) = 2x^3 + 18x^2 + 13x + 3.$$

Since $a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca)$, we can find the result by finding the elementary symmetric polynomials on the roots. Here, we have

$$\frac{1}{\alpha-2} + \frac{1}{\beta-2} + \frac{1}{\gamma-2} = -9$$

and

$$\frac{1}{\alpha-2}\frac{1}{\beta-2} + \frac{1}{\beta-2}\frac{1}{\gamma-2} + \frac{1}{\gamma-2}\frac{1}{\alpha-2} = \frac{13}{2},$$

so the desired sum is $(-9)^2 - 2\frac{13}{2} = 81 - 13 = 68$.

6. Assume that $f(a+b) = f(a) + f(b) + ab$, and that $f(75) - f(51) = 1230$. Find $f(100)$.

Solution: One has that $f(0) = f(0) + f(0)$, so $f(0) = 0$. Moreover, $f(n+1) = f(n) + f(1) + n$, so that $f(n) = \sum_{i=0}^{n-1}(i + f(1)) = \frac{n(n-1)}{2} + nf(1)$. Plugging in $n = 75$ and $n = 51$, one gets $2775 - 1275 + 24f(1) = 1230$, so $f(1) = -\frac{45}{4}$. Thus, $f(100) = 3825$.

7. The expression $\sin 2^\circ \sin 4^\circ \sin 6^\circ \cdots \sin 90^\circ$ is equal to $p\sqrt{5}/2^{50}$, where p is an integer. Find p.

Solution: 192. Let ω be the root of unity $e^{2\pi i/90}$, so we have

$$\prod_{n=1}^{45} \sin(2n^\circ) = \sum_{n=1}^{45} \frac{\omega^n - 1}{2i\omega^{n/2}}.$$

By the symmetry of the sine (and the fact that $\sin(90^\circ) = 1$),

$$\prod_{n=1}^{45} \sin(2n^\circ) = \prod_{n=46}^{89} \sin(2n^\circ),$$

so

$$\left|\prod_{n=1}^{45} \sin(2n^\circ)\right|^2 = \sum_{n=1}^{89} \frac{|\omega^n - 1|}{2} = \frac{90}{2^{89}},$$

where we've used the usual geometric series sum for roots of unity. The product is clearly positive and real, so it is equal to

$$\frac{\sqrt{45}}{2^{44}} = \frac{3\sqrt{5}}{2^{44}},$$

implying that $p = 3 \cdot 2^6 = 192$.

8. Let p be a polynomial with integer coefficients such that $p(15) = 6$, $p(22) = 1196$, and $p(35) = 26$. Assume that $p(n) = n + 82$ for some integer n. Find n.

Solution: 28. Since $p(n) - n - 82 = 0$, the polynomial $p(x) - x - 82$ must factor to $(x - n)q(x)$, where $q(x)$ is another polynomial. The polynomial q will have integer coefficients, because $p(x) - x - 82 = p(x) - p(n) + n - x$, so if we let $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_jx^j$, we get

$$p(x) - x - 82 = (a_1 - 1)(x - n) + a_2(x^2 - n^2) + \cdots + a_j(x^j - n^j).$$

Dividing through by $x - n$ clearly leaves a polynomial with integer coefficients, since $x^i - n^i$ is always divisible by $x - n$. In particular, therefore, $q(15)$ and $q(35)$ are integers, so plugging in 15 and 35 we get that $15 - n$ is divisible by $p(15) - 15 - 82 = -91$ and $35 - n$ is divisible by $p(35) - 35 - 82 = -91$. Since the factors of 91 are just ±1, ±7, and ±13, we must have either $15 - n = -13 \implies n = 28$ or $15 - n = -7 \implies n = 22$. The latter case is ruled out because $p(22) = 116 \neq 22 + 82$.