PUMaC 2010

Algebra A

1. Find the sum of the coefficients of the polynomial $(63x-61)^4$.
2. Calculate $\sum_{n=1}^{\infty}\left(\lfloor\sqrt[n]{2010}\rfloor-1\right)$ where $\lfloor x\rfloor$ is the largest integer less than or equal to x.
3. Let S be the sum of all real x such that $4^x=x^4$. Find the nearest integer to S.
4. Define $f(x)=x+\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}$. Find the smallest integer x such that $f(x)\geq 50\sqrt{x}$.
5. Let $f(x)=3x^3-5x^2+2x-6$. If the roots of f are given by α, β, and γ, find
$$\left(\frac{1}{\alpha-2}\right)^2+\left(\frac{1}{\beta-2}\right)^2+\left(\frac{1}{\gamma-2}\right)^2.$$
6. Assume that $f(a+b)=f(a)+f(b)+ab$, and that $f(75)-f(51)=1230$. Find $f(100)$.
7. The expression $\sin 2^\circ \sin 4^\circ \sin 6^\circ \cdots \sin 90^\circ$ is equal to $p\sqrt{5}/2^{50}$, where p is an integer. Find p.
8. Let p be a polynomial with integer coefficients such that $p(15)=6$, $p(22)=1196$, and $p(35)=26$. Find an integer n such that $p(n)=n+82$.