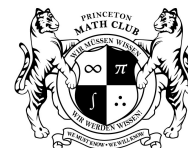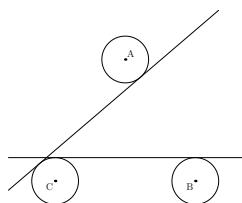# PUMaC 2008-9

Geometry

1. (2 points) If a rectangle's length is increased by 30% and its width is decreased by 30%, by what percentage does its area change? State whether the area increases or decreases.

2. (2 points) What is the area of a circle with a circumference of 8?

3. (3 points) Consider a convex polygon $\mathcal{P}$ in space with perimeter 20 and area 30. What is the volume of the locus of points that are at most 1 unit away from some point in the interior of $\mathcal{P}$?

4. (3 points) A cube is divided into 27 unit cubes. A sphere is inscribed in each of the corner unit cubes, and another sphere is placed tangent to these 8 spheres. What is the smallest possible value for the radius of the last sphere?

5. (4 points) Two externally tangent circles have radius 2 and radius 3. Two lines are drawn, each tangent to both circles, but not at the point where the circles are tangent to each other. What is the area of the quadrilateral whose vertices are the four points of tangency between the circles and the lines?

6. (4 points) Circles A, B, and C each have radius r, and their centers are the vertices of an equilateral triangle of side length $6r$. Two lines are drawn, one tangent to A and C and one tangent to B and C, such that A is on the opposite side of each line from B and C. Find the sine of the angle between the two lines.

PUMaC 2008-9
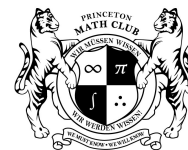
Geometry

7. (5 points) How many ordered pairs of real numbers (x, y) are there such that $x^2 + y^2 = 200$ and

$$\sqrt{(x-5)^2 + (y-5)^2} + \sqrt{(x+5)^2 + (y+5)^2}$$

is an integer?

8. (5 points) Infinitesimal Randall Munroe is glued to the center of a pentagon with side length 1. At each corner of the pentagon is a confused infinitesimal velociraptor. At any time, each raptor is running at one unit per second directly towards the next raptor in the pentagon (in counterclockwise order). How far does each confused raptor travel before it reaches Randall Munroe?

9. (7 points) Let $\mathcal{H}$ be the region of points (x, y), such that $(1, 0)$, (x, y), $(-x, y)$, and $(-1, 0)$ form an isosceles trapezoid whose legs are shorter than the base between (x, y) and $(-x, y)$. Find the least possible positive slope that a line could have without intersecting $\mathcal{H}$.

10. (7 points) A cuboctahedron is the convex hull of (smallest convex set containing) the 12 points $(\pm 1, \pm 1, 0)$, $(\pm 1, 0, \pm 1)$, $(0, \pm 1, \pm 1)$. Find the cosine of the solid angle of one of the triangular faces, as viewed from the origin. (Take a figure and consider the set of points on the unit sphere centered on the origin such that the ray from the origin through the point intersects the figure. The area of that set is the solid angle of the figure as viewed from the origin.)