PUMaC 2008-9

Algebra

1. (2 points) Solve for x: $x = 2 + \frac{4(2^6)}{11-3}$

2. (2 points) What is $3(2\log_4(2(2\log_3 9)))$?

3. (3 points) Given the sequence 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, ..., find n such that the sum of the first n terms is 2008 or 2009.

4. (3 points) Find the product of the minimum and maximum values of $\frac{3x+1}{9x^2+6x+2}$.

5. (4 points) How many real roots do $x^5 + 3x^4 - 4x^3 - 8x^2 + 6x - 1$ and $x^5 - 3x^4 - 2x^3 + 10x^2 - 6x + 1$ share?

6. (4 points) Let $f(n) = 9n^5 - 5n^3 - 4n$. Find the greatest common divisor of $f(17)$, $f(18)$, ..., $f(2009)$.

7. (5 points) What's the greatest integer n for which the system $k < x^k < k+1$ for $k = 1, 2, \ldots, n$ has a solution?

8. (5 points) Let $H_k = \sum_{i=1}^{k} \frac{1}{i}$ for all positive integers k. Find an closed-form expression for $\sum_{k=1}^{n} H_k$ in terms of n and H_n.

9. (7 points) Find the polynomial f with the following properties:
   - its leading coefficient is 1,
   - its coefficients are nonnegative integers,
   - $72 | f(x)$ if x is an integer,
   - if g is another polynomial with the same properties, then $g - f$ has a nonnegative leading coefficient.

10. (7 points) If $p(x)$ is a polynomial with integer coefficients, let $q(x) = \frac{p(x)}{x(1-x)}$. If $q(x) = q\left(\frac{1}{1-x}\right)$ for every $x \neq 0$, and $p(2) = -7$, $p(3) = -11$, find $p(10)$.