PUMaC 2008-9

Combinatorics A

1. (2 points) How many 3-digit numbers contain the digit 7 exactly once?

2. (2 points) Draw a regular hexagon. Then make a square from each edge of the hexagon. Then form equilateral triangles by drawing an edge between every pair of neighboring squares. If this figure is continued symmetrically off to infinity, what is the ratio between the number of triangles and the number of squares?

3. (3 points) Evaluate
$$\sum_{m=0}^{2009} \sum_{n=0}^{m} \binom{2009}{m} \binom{m}{n}.$$

4. (3 points) Find the sum of the values of x for which
$$\binom{x}{0} - \binom{x}{1} + \binom{x}{2} - \cdots + \binom{x}{2008} = 0.$$

5. (4 points) In how many ways can Alice, Bob, Charlie, David, and Eve split 18 marbles among themselves so that no two of them have the same number of marbles?

6. (4 points)
$$\begin{array}{l} xxxx \\ xx \\ x \\ x \end{array}$$
In how many ways can you fill in the xs with the numbers 1-8 so that for each x, the numbers below and to the right are higher.

7. (5 points) Joe makes two cubes of sidelengths 9 and 10 from 1729 randomly oriented and randomly arranged unit cubes, which are initially unpainted. These cubes are dipped into white paint. Then two cubes of sidelengths 1 and 12 are formed from the same unit cubes, again randomly oriented and randomly arranged, and these cubes are dipped into paint remover. Joe continues to alternately dip cubes of sides 9 and 10 into paint and cubes of sides 1 and 12 into paint remover ad nauseam. What is the limit of the expected number of painted unit cube faces immediately after dipping in paint remover?

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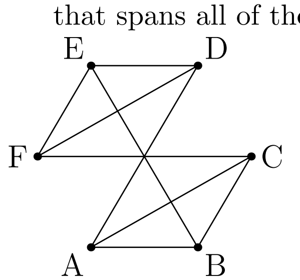
8. (5 points) SET cards have four characteristics: number, color, shape, and shading, each of which has 3 values. A SET deck has 81 cards, one for each combination of these values. A SET is three cards such that, for each characteristic, the values of the three cards for that characteristics are either all the same or all different. In how many ways can you replace each SET card in the deck with another SET card (possibly the same), with no card used twice, such that any three cards that were a SET before are still a SET?

(Alternately, a SET card is an ordered 4-tuple of 0s, 1s, and 2s, and three cards form a SET if their sum is $(0, 0, 0, 0)$ mod 3; for instance, $(0, 1, 2, 2)$, $(1, 0, 2, 1)$, and $(2, 2, 2, 0)$ form a SET. How many permutations of the SET cards maintain SET-ness?) 9. (7 points) How many spanning trees

does the following graph (with 6 vertices and 9 edges) have? (A spanning tree is a subset of edges that spans all of the vertices of the original graph, but does not contain any cycles.)



10. (7 points) In his youth, Professor John Horton Conway lived on a farm with 2009 cows. Conway wishes to move the cows from the negative x axis to the positive x axis. The cows are initially lined up in order $1, 2, \ldots, 2009$ on the negative x axis. Conway can give two possible commands to the cows. One is the PUSH command, upon which the first cow from the negative x axis moves to the lowest position on the positive y axis. The other is the POP command, upon which the cow in the lowest position on the y axis moves to the positive x axis. For example, if Conway says PUSH POP 2009 times, then the resulting permutation of cows is the same, $1, 2, \ldots, 2009$. If Conway says PUSH 2009 times followed by POP 2009 times, the resulting permutation of cows is $2009, \ldots, 2, 1$. How many output permutations are possible after Conway finishes moving all the cows from the negative x axis to the positive x axis?