PUMaC 2008-9

Algebra

1. (2 points) Given the sequence 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, . . ., find n such that the sum of the first n terms is 2008 or 2009.

2. (2 points) What is the polynomial of smallest degree that passes through $(-2, 2)$, $(-1, 1)$, $(0, 2)$, $(1, -1)$, and $(2, 10)$?

3. (3 points) Let $f(n) = 9n^5 - 5n^3 - 4n$. Find the greatest common divisor of $f(17)$, $f(18)$, . . . , $f(2009)$.

4. (3 points) What's the greatest integer n for which the system $k < x^k < k + 1$ for $k = 1, 2, \ldots, n$ has a solution?

5. (4 points) Let $H_k = \sum_{i=1}^{k} \frac{1}{i}$ for all positive integers k. Find an closed-form expression for $\sum_{k=1}^{n} H_k$ in terms of n and H_n.

6. (4 points) Let x be the largest root of $x^4 - 2009x + 1$. Find the nearest integer to $\frac{1}{x^3-2009}$.

7. (5 points) Suppose $x^9 = 1$ but $x^3 \neq 1$. Find a polynomial of minimal degree equal to $\frac{1}{1+x}$.

8. (5 points) Find the polynomial f with the following properties:
   - its leading coefficient is 1,
   - its coefficients are nonnegative integers,
   - $72|f(x)$ if x is an integer,
   - if g is another polynomial with the same properties, then $g - f$ has a nonnegative leading coefficient.

9. (7 points) If $p(x)$ is a polynomial with integer coefficients, let $q(x) = \frac{p(x)}{x(1-x)}$. If $q(x) = q\left(\frac{1}{1-x}\right)$ for every $x \neq 0$, and $p(2) = -7$, $p(3) = -11$, find $p(10)$.

10. (7 points) Find the sum of all integer values of n such that the equation $\frac{x}{(yz)^2} + \frac{y}{(zx)^2} + \frac{z}{(xy)^2} = n$ has a solution in positive integers.