PUMaC 2007 Power Test: Lattices

A *real n-dimensional lattice* Λ is a set of n-tuples $(a_1, a_2, \ldots, a_n)$ of real numbers with the following properties:

1) The all-zero-tuple $\mathbf{0} = (0, 0, \ldots, 0)$ belongs to Λ.

2) If $u = (a_1, a_2, \ldots, a_n)$ and $v = (b_1, b_2, \ldots, b_n)$ belong to Λ, then so do $-u = (-a_1, -a_2, \ldots, -a_n)$ and $u + v = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$.

For example, the set $\mathbb{Z}^2$ of ordered pairs of integers forms a lattice, as does the trivial n-dimensional lattice, which consists of the single n-tuple $\mathbf{0}$.

The *distance* between two n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ in a lattice Λ is the standard Euclidean distance $\sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + \ldots + (b_n - a_n)^2}$.

Given a lattice Λ, we refer to its elements as either *points*, or the *vectors* they represent.

For the problems below, show all your work and give justification for all answers, unless otherwise indicated. Answers given without justification will not be given full credit.

1 Minimal Vectors

The *minimal vectors* of a lattice are the ones (other than the 0 vector) represented by the points closest to the origin, $\mathbf{0}$. The *norm* of a minimal vector is the square of the distance from the point that represents it to the origin. The norm of a minimal vector of a lattice is called the *minimal norm* of that lattice. In $\mathbb{Z}^2$, the minimal vectors are $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.

a) How many minimal vectors are there in the 1-dimensional lattice $\mathbb{Z}^1$?
b) How many minimal vectors are there in the 3-dimensional lattice $\mathbb{Z}^3$?
c) Given any integer m, how many minimal vectors are there in the m-dimensional lattice $\mathbb{Z}^m$?

The *checkerboard lattice* D_n is the set of all points in $\mathbb{Z}^n$ for which the sum of all n coordinates is even.

d) How many minimal vectors are there in D_3?
e) Given any integer m, how many minimal vectors are there in D_m?

2 Bases

Consider the lattice $\mathbb{Z}^3$. We can find three vectors v_1, v_2, v_3 such that any vector in $\mathbb{Z}^3$ may be expressed in the form $\sum_{i=1}^{3} k_i v_i$, where k_1, k_2, k_3 are integers. The vectors v_1, v_2, v_3 are then called a *basis* for the lattice $\mathbb{Z}^3$. The vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ form a basis for $\mathbb{Z}^3$, because any vector $(a,b,c) \in \mathbb{Z}^3$ can be expressed as $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$, and the coefficents a, b, and c are integers.

a) Give a basis for $\mathbb{Z}^m$.
b) Give a basis for D_m.
c) Consider the 8-dimensional lattice with basis vectors

$$\begin{array}{cccccccc}
(2, & 0, & 0, & 0, & 0, & 0, & 0, & 0), \\
(-1, & 1, & 0, & 0, & 0, & 0, & 0, & 0), \\
(0, & -1, & 1, & 0, & 0, & 0, & 0, & 0), \\
(0, & 0, & -1, & 1, & 0, & 0, & 0, & 0), \\
(0, & 0, & 0, & -1, & 1, & 0, & 0, & 0), \\
(0, & 0, & 0, & 0, & -1, & 1, & 0, & 0), \\
(0, & 0, & 0, & 0, & 0, & -1, & 1, & 0), \\
(\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}).
\end{array}$$

Which points can be in this lattice, what is the minimal norm, and how many minimal vectors are there?

The lattice given in part (c) is called the E_8 *diamond lattice*.

d) Prove that we can describe the lattice $\mathbb{Z}^3$ with D_3 as follows: $\mathbb{Z}^3$ consists of all points in D_3 and all points which are obtained by adding the vector $(1,1,1)$ to a point in D_3. (We shall denote this by $\mathbb{Z}^3 = D_3 \cup (D_3 + (1,1,1))$.)
e) Give a similar description of E_8 in terms of D_8.

3 Sphere Packings

Lattices can be useful for describing *sphere packings*, the classical problem of finding how densely a large number of identical spheres can be packed together, wasting as little space as possible. The sphere-packing problem can be generalized to any number of dimensions. In two dimensions, we have circle-packing. In 500 dimensions, we have the packing of 500-dimensional hyperspheres. The *density* of a packing is defined as the fraction of the total volume (or area, or n-dimensional volume) occupied by the spheres. For the purposes of this problem, packings will always tessellate n-dimensional space, so their density can be calculated by analyzing one tessella. If the centers of the spheres of a packing form a lattice, we say that a basis for this lattice is also a basis for the packing. For simplicity, we will consider only packings of n-dimensional space extending infinitely in all directions, not of containers with boundaries.

a) Draw a diagram of the densest (2-dimensional) circle packing. No justification is necessary.

If we place the center of one circle at the origin, the centers of the circles in the packing from part (a) form a lattice which we call A_2. We will give the name A_2 to any lattice having this shape (so the lattice is only unique up to rotation and scale about the origin).

b) Find a basis for your packing from part (a) if your circles have radius $\frac{1}{2}$ and lie in the xy-plane.
c) If the circles from part (a) have radius $\frac{\sqrt{2}}{2}$ and are placed on the plane $x+y+z=0$ in $\mathbb{R}^3$, prove that $(1,-1,0)$ and $(0,1,-1)$ form a basis for the packing, up to rotation.
d) Find the density of the packing in parts (b) and (c).
e) The densest sphere-packing in three dimensions is believed (but not proven) to be one represented by the lattice D_3. Compute the maximum density of this packing.

4 Sub-Lattices of E_8

Two n-dimensional vectors $(v_1, v_2, \ldots, v_n)$ and $(u_1, u_2, \ldots, u_n)$ are *perpendicular* if $\sum_{i=1}^{n} v_i u_i = 0$. For example, the 5-dimensional vectors $(1,4,-3,2,8)$ and $(-1,1,3,-5,2)$ are perpendicular because $1\cdot(-1)+4\cdot 1+(-3)\cdot 3+2\cdot(-5)+8\cdot 2=0$.

a) Given a vector $v \in E_8$, the set of vectors in E_8 that are perpendicular to v form the lattice

E_7. Taking $v = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, describe the resulting lattice E_7, giving necessary and sufficient conditions for a point to be in the lattice, as well as the minimal norm and the number of minimal vectors.

Any n-dimensional lattice L_n has a dual lattice L_n* consisting of all n-dimensional vectors $(x_1, x_2, \ldots, x_n)$ such that $\sum_{i=1}^{n} x_i u_i$ is an integer for every vector $(u_1, u_2, \ldots, u_n) \in L_n$.

b) One of the minimal vectors in E_7* is $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}, -\frac{3}{4})$. How many minimal vectors are there in E_7*?

Given a subset Λ of E_8 that forms a (rotated) lattice A_2, all the vectors in E_8 that are perpendicular to every vector in Λ form the lattice E_6.

c) Prove that $(1, 0, 0, 0, 0, 0, 0, 1)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ form a basis for A_2.
d) Using the version of A_2 from (c), find the minimal vectors of E_6.
e) Find the minimal vectors of E_6*.

5 Complex Lattices

A complex lattice is a lattice with complex coordinates instead of real coordinates. A *complex n-dimensional Gaussian lattice* Λ has a basis $v_1, v_2, \ldots, v_n$ of vectors with complex coordinates, and any vector in Λ may be expressed in the form $\sum_{i=1}^{n} k_i v_i$, where $k_1, k_2, \ldots, k_n$ are *Gaussian integers*. The Gaussian integers $\mathcal{G}$ are the numbers of the form $a + bi$, where a and b are integers.

The ordered pair (x, y) can be used to represent the complex number $x + yi$, so any $2n$-dimensional real lattice can be expressed as an n-dimensional complex lattice. For example, the lattice $\mathbb{Z}^2$ is the complex lattice $\mathcal{G}^1$.

a) Express the real lattice $\mathbb{Z}^{2m}$ as an m-dimensional complex lattice over the Gaussian integers.

Another sort of complex lattice can be defined over the *Eisenstein integers* $\mathcal{E}$ (instead of the Gaussian integers), the set of numbers of the form $a + b\omega$, where a and b are real integers and $\omega = \frac{-1+\sqrt{3}i}{2}$. (Note that $\omega^3 = 1$.) The Eisenstein integers are useful for expressing hexagonal and diamond lattices as complex lattices. For example, the lattice A_2 can be simply expressed as the complex lattice $\mathcal{E}^1$.

b) Consider the complex 3-dimensional Eisenstein lattice with basis vectors $(\sqrt{3}i, 0, 0)$, $(1, -1, 0)$, and $(1, 0, -1)$. Find the minimal vectors of this lattice, and show that the lattice is equivalent to E_6* (ie., that it has the same properties as the lattice found in part (e) of question 4).