Judge's Commentary: The Outstanding Tollbooths Papers

Kelly Black
Dept. of Mathematics
Union College
Schenectady, NY 12308
blackk@union.edu

Overview of the Problem

The teams were asked to examine the flow of traffic through a toll plaza. One of the difficulties in this problem is that the traffic flow through a toll plaza is not actively managed; rather, the traffic through the system is passively managed by the careful design of the roadway and the toll-collection system. On most roadways, a toll plaza consists of a long transition to an increased number of lanes, the collection system (the tollbooths), and a long transition back to the original number of lanes. Interestingly, only a very small number of entries suggested adding an active management component to the toll plaza, such as ways to restrict the way that drivers can change lanes.

In the last paragraph of the problem statement three tasks/questions were given:

- "Make a model to help determine the optimal number of tollbooths to deploy in a barrier-toll plaza."
- "Explicitly consider the scenario where there is exactly one tollbooth per incoming lane."
- "Under what conditions is this more or less effective than the current practice?"

At first glance, it would seem that the considerable power of queueing theory would be readily available for this problem. Unfortunately, this is only true to a certain extent. The nature of a toll plaza is that multiple lanes expand into even more lanes into which people can change if they feel it is advantageous. Once again, people get in the way of good mathematics.

The UMAP Journal 26 (3) (2005) 391–399.

On second thought, however, the difficulty of the problem should not be a surprise. As far as we are aware, no state's Dept. of Transportation has been able to get this problem right. Moreover, traffic designers often only have one chance to get it right. The cost of making changes to an existing toll plaza can be prohibitive.

In the following section, we provide an overview of the kinds of solutions that were submitted, including some comments on the judges' reactions. We also provide an overview of the judging process itself and the challenges that this particular problem represented to the judges. Finally, some tips and pointers are provided in reaction to some of the things that appeared in many of the team's submissions.

The Problem at Hand

Modeling a Toll Plaza

The first of the three questions required teams to create a mathematical model of a barrier-toll plaza. The most common mathematical model treated the toll plaza as a queue. Unfortunately, the complex nature of the toll plaza is not easily described as a simple queue. For example, the lanes diverge into more lanes at the entrance, and drivers are able to change lanes. Also, the lanes must combine again after the service area, and crowded traffic after the tollbooth can impact the traffic in the entrance of the toll plaza.

Of the entries that modeled a toll plaza using queueing theory, what set them apart was how they handled the various subtleties of a toll plaza. For example, teams had to make decisions on how cars move in the entrance and exit sections of a toll plaza. Teams also had to decide on how each car is handled by the tollbooth attendants. For example, some teams assumed that each car could be handled in the same amount of time. Other teams assumed that the time required for each car was a random variable with a prescribed probability distribution. To make this decision even more difficult, the teams had to decide how to handle the various payment methods available, such as cash, EZ-Pass, or other remote rapid-payment methods.

Of those teams that assumed that the service time required for each car varied, some treated the entrance section as a queue and the actual tollbooth as a separate queue. In this situation, the resulting chain of queues could be coupled and described if the two probability distributions were similar. For example, the majority of such entries assumed that the distributions of the cars entering the system and tollbooth service times were both Poisson with different parameters.

While the more advanced entries were able to bring together the entrance and the tollbooths in the plaza, the exit represented a significant difficulty. The majority of entries briefly discussed the potential problems of the traffic after the service booths but did not include the effects of the exits in their models.

In fact, the teams that did explicitly consider the exit areas usually only did so in the context of a simulation in a computational model.

In addition to the queueing theory approach, a less common approach was to model the flow of traffic as a fluid. The resulting models were much more complex than those based on queueing theory. The process of matching the physical situation to a flow and then converting the results back to what is happening within a toll plaza represented a substantial difficulty for those taking this approach.

Besides the construction of a mathematical model, the most popular approach to this problem made use of simulations based on a computational model. Such models were usually based on either a cellular automata model or a highly modified queue making use of a time-stepping scheme allowing for lane changes. The more advanced approaches also factored the exit areas into the simulation.

A computational model required a much different approach to the analysis and discussion of the results. The results are a composite of many runs and take on a statistical nature. Furthermore, the large number of variables—the number of lanes before the plaza, the number of lanes in the plaza, the waiting time, the way cars enter the system, the length of the plaza, and a wide variety of other factors—make it difficult to reach concrete conclusions about the best design for a toll plaza. This is especially true given the short time available to develop the computational model, implement it, decide which situations to use, run it, and examine the results.

The judges took this into consideration and did not expect a complete examination. This approach did require a more complete description of the computational model, though, since the number of assumptions that can be incorporated was significantly greater. There was also a heightened expectation of doing more in the way of a sensitivity analysis with respect to some of the various parameters.

While many different approaches were submitted, the entries that received the higher ratings examined at least two different models. The most common combination by far was a simple queue and a computational model. The most striking aspect of this was that few teams explicitly stated a comparison of the two results under identical circumstances. Those that did stood out, and the results of the comparison helped to establish a good benchmark of the computational model.

One Tollbooth Per Lane

The second question in the problem statement required each team to use their models for a toll plaza that has one tollbooth for each lane of travel on the road. This established a sanity check that each team had to examine. Surprisingly, a number of teams did not examine this situation, which resulted in a penalty for ignoring one of the stated requirements.

What Is "Best"

The final question required the teams to determine best practice in designing a toll plaza—to define a way to compare different configurations of a toll plaza. Each team had to balance the competing costs of each driver's time, the cost of operating the toll plaza, and the cost of construction.

One of the most surprising aspects of this year's competition is that few teams explicitly defined what they thought would be the cost of operating a toll plaza. The vast majority of teams simply compared the average waiting time for the drivers under various circumstances with little comment or justification. Those teams that looked at a nontrivial cost of operating the tollbooths based on the cost of lost productivity of the drivers and the cost of operating the tollbooths certainly stood apart from the others.

The few teams that did examine this part of the problem reported some of the most interesting results. In fact, in some circumstances the option of not building a tollbooth is the most satisfactory option to almost everybody except maybe the tollbooth operators themselves!

Overview of the Judging Process

We give an overview of the judging process, including some general observations about some of the entries submitted for the competition. In the subsections that follow we try to provide some insight into what the judges discussed prior to the actual judging, first impressions of a paper, and some of the small details that help to make an entry stand out.

First, we try to provide a broad overview of the judging process itself. The papers are examined in a two-step process. The first round, or triage round, is the first pass. In this round, judges are able to examine the papers for only a relatively short time. When a judge begins reading a paper in this first stage, the question is, "Should the paper be read in closer detail?" If the answer is "yes" or even "maybe," then it is passed on to the second stage. Because we try to give each paper the benefit of the doubt, it is difficult to state what necessary essence is required to move past this round.

At the most basic level, the quality of the writing and the consistency of the summary with the rest of the paper is vital in this first stage. It is a really bad idea to make a judge work too hard on a paper. The easier it is for a judge to determine how the students interpreted the problem, the approach used, and the results, the easier it is for the judges to determine the quality of the work.

Entries that remain through the second stage are given much closer, detailed readings. For example, papers that remain on the final day of judging are read by as many as eight different judges for at least an hour per entry. During this time the judges sometimes confer with one another if they are not sure about an equation, result, or the wording in a section. For the most part, though, each judge tries to provide an independent review of the student's work.

Discussion Before Judging Began

Before the judging began, the judges got together to discuss the problem. As usual, the problem was nontrivial, and we judges had to ensure that we understood what was being asked. The judges had to carefully parse the original question. For example, this year the problem included some very specific tasks that were given in the last paragraph of the problem, and whether or not an entry specifically addressed those questions was important.

Additionally, each judge initially read through a set of randomly chosen papers. In the second stage of the process, this set of papers was adjusted to ensure that each judge viewed papers with a wide variety of initial scores. The purpose of this protocol is to make sure that we also took into consideration how the various teams interpreted the question and how they reacted to the problem.

After these initial readings, the judges had to agree on what was important and how to provide a consistent mechanism for comparing different entries. Each year, the relative importances of the various aspects of a paper are tailored to the particular problem; but in general, the kinds of things that judges look for in a paper are relatively consistent. This year the judges decided that the following aspects were important:

Summary This is the first thing that a judge sees. A summary should provide a brief overview of the problem, a brief review of the methodologies used, and an overview of the conclusions. It is a difficult challenge to include all these things on one page and do it well, but it does provide the first impression.

Assumptions and Justifications In constructing a mathematical representation of a physical system some simplifications must be made and some subtleties must be left out. The parts of the problem deemed most important—as well as what is left out of the model—must be made explicit.

Model/Analysis One of the novel aspects of this year's problem was the close association between the mathematical model and the analysis of the problem. The stochastic nature of the problem, as well as the prevalence of entries making use of both queuing theory and simulation, made it difficult to separate these two aspects of the problem. In the end, the judges decided not to treat them separately, so that it would be easier to compare entries whose balance varied between the different approaches used.

Results/Validation It is not unusual to see many papers that make use of a variety of approaches and techniques, but this problem resulted in more entries than usual that employed at least two solution techniques. It was more important than ever for the teams to be able to compare the different results as well as interpret the results.

Sensitivity Between the stochastic nature of the problem and the wide use of simulation, the validation of the results had to include some way to test the robustness of the conclusions. One of the most important ways to do this is to examine what happens after changing the values of certain parameters or changing the assumptions on what kind of probability distributions to use. For example, some teams that used queuing theory examined their conclusions under a variety of assumptions about the time required by an individual tollbooth attendant to complete one transaction. If a small variation in the service time resulted in a large change in the average waiting times, that is an indication that the conclusions may be circumspect.

Strength & Weakness Any mathematical model requires many assumptions and simplifications and is only good for a restricted situation. It is vital that the modeler identify the conditions under which the mathematical model is appropriate. Each team was expected to demonstrate explicitly that they had done some critical analysis of the model itself and to identify what they felt was good and bad about the model.

Clarity/Communication One of the key aspects of any problem is to be able to share the results. The methods employed, the results that are delivered, and the analysis of both the methods and the results must be clearly described. This is the filter through which all mathematics is shared.

Communicating Mathematics

As mathematicians, we are engaged in a social exercise that absolutely requires us to share our work with one another, however sharing mathematical ideas can be extremely difficult. In fact, it is difficult enough that we often try very hard to avoid putting our students through the difficult learning process associated with writing and sharing mathematical ideas. We often have our students take part in writing proofs or problem solving, but putting it all together in a formal report or a paper the first time can be an excruciating process.

There are many excellent books and other resources for students that offer a better introduction than can be expressed here. In fact, from the many excellent entries it is clear that those resources are being exploited. We focus on just a few general issues that stand out in this year's event.

- Some teams presented a *narrative of the team's activities*. An entry that lists how the team approached the problem and chronicles what the team did (or tried to do) puts the team at a severe disadvantage. In contrast, an entry in the format of a self-contained report immediately stands out. In such a report, the problem is restated, including the results; the various methodologies that are used are clearly stated; the analyses are given; and the results are clearly stated, including a critical examination of the approach and the results.

- One aspect of writing that even advanced writers struggle with is graphing. When a plot is provided, it should be clearly introduced and described in the text, including a proper reference to the figure number. It should be clear to the reader from the text of the report what to look for in the plot before turning to look at the plot. This year's problem is an excellent example of the importance of describing plots and figures. Providing a graphical example of cars moving through a toll plaza over time is difficult and each team attempted to do this is in many different ways. Furthermore, some teams provided sequences of figures to demonstrate a particular transition, and there is a huge burden on the writers to explain what the reader should be looking for and what the implications are.

In general, when a figure or plot is provided, the text should provide a detailed explanation of what is in the figure. Also, the caption and labels in the figure should succinctly describe what the figure is. Of course, the axes should be labeled and units should be provided. One thing that was different for this year, however, was that the judges gave teams more leeway in how discrete vs. continuous functions were displayed. Discrete data should be displayed as discrete and not with lines drawn between points. This year was different in that many figures represented the organization of the cars in a queue that might be discrete according to the model even though the domain (time) could be continuous.

- Some of the entries included many tables. Almost everything said above for figures also applies to tables. Tables should be clearly labeled and explained in detail in the text of the report. The easier you make it for a judge to determine what is in a table and why it is important, the more likely the judge will be happy. A happy judge is a higher-scoring judge!

- Finally, a small thing that is very likely to keep an otherwise good paper from being held back in the early rounds. Some teams have a difficult time integrating equations and citations within the text of the report. Equations and citations should be correctly integrated into each sentence using proper grammar. Some teams that do excellent work make it extremely difficult for themselves when the equations or citations are set apart from each sentence and not properly integrated into the flow of the text.

The Little Things

The vast majority of teams do great work, and it is always exciting to see what the teams are able to accomplish in such a short amount of time. It is also important to be able to share and express the ideas developed by each team in a formal report. This final product is the vehicle used to communicate the team's ideas and techniques. It is not a narrative of what the team did but is an opportunity to educate and persuade others to follow up on the team's excellent work.

There are a number of simple things that can be done to make an entry easier to read. Some of these may appear to be trivial, but they make the judges task easier which in turn makes it easier for the judges to concentrate on the ideas rather than the way the ideas are expressed:

Strengths & Weaknesses This aspect of an entry demonstrates whether or not a team has provided a critical inquiry into the methods and techniques developed. Including this aspect as a separate section of a report makes it much easier for the judges to identify easily this important aspect of the team's efforts.

Table of Contents Given the growing number of teams that use LaTeX, it is shocking how few entries provide a table of contents. This is a trivial step that can radically improve the readability of a report.

Citations A paper that makes ample use of citations properly and consistently integrated into the text is guaranteed to stand out. For example, many papers included citations when providing the definitions of functions describing the way cars entered the toll plaza but failed to provide a citation when stating some of the results that happen to come directly from the relevant literature.

Equation Numbers Number all equations even if they are not explicitly referred to in the text. This makes it easier for judges to confer when there is a question about a particular equation.

Units Units are important. Always make sure that the definition of a variable, parameter, or function includes its units. Also, always check a result to make sure that the units are correct. This is one of the first checks judges make when confronted with a result that is not obvious. (Always keep in mind the difference between a quantity and its rate of change!)

Conclusions

Each year we are amazed at the high quality of the entries. The things that the teams can accomplish in a weekend are a testament to the quality of the team's training and hard work. The teams receiving the higher honors should be proud to stand out in such an incredible field.

This year the teams that submitted entries for the Tollbooth Problem focused their efforts on the optimal design of a toll plaza. They had to consider the number of lanes, the lengths of lane transitions, the times required to collect the fares, and a wide variety of other factors. Most of the teams made use of either queuing theory, comparisons to fluid flow, or simulation via a computational model. The teams that received higher recognition from the judges derived more than one model and made comparisons among their models.

One aspect that set entries apart was the analysis and critical evaluation of their models and results. As usual, a sensitivity analysis of the models is important; but because of the nature of this year's problem, an even higher value was placed on this important aspect of the modeling process.

Finally, each team's entry consists of a written report designed to educate and persuade. This is a difficult task in itself, but the teams are asked to do this without the benefit of an outside editing process; they must somehow build editing into their efforts themselves. Adding to the difficulties, good writing is most effective when it does not get in the way of the ideas that the writer is trying to convey and is not noticed until after the reader looks back and realizes what has been shared.

About the Author

Kelly Black is a faculty member in the Dept. of Mathematics at Union College. He received his undergraduate degree in Mathematics and Computer Science from Rose-Hulman Institute of Technology and his Master's and Ph.D. from the Applied Mathematics program at Brown University. His research is in scientific computing and he has interests in computational fluid dynamics, laser simulations, and mathematical biology.