Stanford University Mathematics Camp (SUMaC) 2019 Admission Exam

- Solve as many of the following problems as you can. Your work on these problems together with your grades in school, teacher recommendations, and answers to questions on the application form are all used to evaluate your SUMaC application. Although SUMaC is very selective with a competitive applicant pool, correct answers to every problem are not required for admission.
- There is no time limit for this exam other than the application deadline of March 13.
- Feel free to report partial progress toward a solution, in the event you are unable to solve a problem completely.
- Please include clear, detailed explanations with ***all of your answers***; numerical answers or formulas with no explanation are not useful for evaluating your application.
- None of these problems require a calculator or computer.
- You are expected to do your own work without the use of any outside source (books, internet search, etc). If you recognize one of the problems from another source, or if you receive any assistance, please indicate this on your solution.
- ***Please do not share these problems or your solutions with anyone.***

1. Find all positive integers x and y that satisfy $x^2 - y^2 = 2019$? Are there any positive integers x and y such that $x^3 - y^3 = 2019$? Explain.

2. Find a polynomial $p(x)$ with integer coefficients for which $\sqrt{7} - \sqrt{3}$ is a root. That is find $p(x)$ such that for some non-negative integer n, and integers $a_0, a_1, a_2, \dots a_n$,

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

and $p(\sqrt{7} - \sqrt{3}) = 0$.

3. Consider the following tile puzzle. You have a 2×3 square grid, and on the grid you have five square tiles numbered 1, 2, 3, 4, and 5 filling all but the bottom-right square (referred to the current *empty square*). The *starting position* is the following arrangement

A move in this game consists of sliding a tile into the empty square from a square adjacent to the empty square (leaving its previous location empty). No tile can leave the grid whether by lifting it up or sliding it off the sides. Is it possible to obtain the following configuration from the starting position in a series of moves? Explain.

4. An equilateral has sides of length 1 cm.

   (a) Show that for *any configuration* of five points on this triangle (on the sides or in the interior), there is at least one pair of from these five points such that the distance between the two points in the pair is less than or equal to .5 cm.
   (b) Show that .5 (in part (a)) cannot be replaced by a smaller number even if there are 6 points.
   (c) If there are eight points, can .5 be replaced by a smaller number? Prove your answer.

5. Suppose n is a positive integer. The (imaginary) sea of Babab has islands each of which has an n-letter name that uses only the letters "a" and "b," and such that for each n-letter name that uses only the letters "a" and "b," there is an island. For example, if $n = 3$, then Aaa, Aab, Aba, Baa, Abb, Bab, Bba and Bbb are the islands in the sea of Babab. The transportation system for Babab consists of ferries traveling back and forth between each pair of islands that differ in exactly one letter. For example, there is a ferry connecting Bab and Bbb since they differ only in the second letter.

   a) How many islands and how many ferry routes are there in terms of n? Count the ferry route for both directions as a single ferry route, so for example, the ferry from Bab to Bbb is the same ferry route.

   Babab does not have much in the way of natural resources or farm land so nearly all food and supplies are provided by the Babab All Bulk Company (BABCO). The people of Babab (Bababians) desire easy access to a BABCO store, where "easy access" means there is a BABCO store on their own island or on one that they can get to with a single ferry ride. However, BABCO finds it uneconomical to give the people on one island easy access to two different BABCO stores, and BABCO is willing to deny some Bababians easy access to a BABCO store in order to meet this restriction.

   b) In the cases $n = 3$, $n = 4$, and $n = 5$, what is the maximum number of stores that BABCO can build while satisfying the restriction than *no one has easy access to more than one BABCO store*? Be sure to prove your answer is optimal.

   c) Now suppose BABCO changes its strategy and decides it wants to be sure *every Bababian has access to a BABCO store even if it means some Bababians have easy access to two stores*. What is the minimum number of stores needed to satisfy this condition in the cases $n = 3$, $n = 4$, and $n = 5$?

   d) Can you find optimal solutions to parts b and c for $n = 6$?

6. Suppose you have a 10×10 checkerboard and a deck of 2×2 cards with squares that match the size of the squares of the checkerboard, so 25 of these cards can be used to completely cover the checkerboard. If we allow the cards to overlap each other, there are many ways to cover the checkerboard. We say that an arrangement of cards is a *covering* if all of the cards in the arrangement are lined up with the squares on the checkerboard, and they are completely on the checkerboard, possibly overlapping, and every square of the checkerboard has at least one card on top of it. We call a covering of the checkerboard *redundant* if one of the cards can be removed and the checkerboard is still covered. A covering of the checkerboard is *non-redundant* if it is no longer a covering if any card is removed. Clearly, the smallest non-redundant covering has 25 cards. The aim of this problem is to find bounds on the number of cards in the largest possible non-redundant covering.

   a) Show that there is a non-redundant covering with 35 cards.
   b) Show that every covering with 55 cards is redundant.
   c) Can you improve these bounds?