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第二页为创新性申明

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MONOTONICITY FORMULA ON CIGAR SOLITON

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Abstract. In this paper, we prove a monotonicity formula on cigar soliton. Using the monotonicity we obtain a three-ball type theorem.

keywords: monotonicity, cigar soliton, three-ball theorem

1. Introduction

Monotonicity formula on cigar soliton contains profound geometrical significance and would help prove the three-ball theorem on it. Many people studied similar type inequalities, such as [2], [3], [4], [5], [7], [8]. In the paper[2], Tobias H. Colding, Camillo De Lellis, and William P. Minicozzi II use monotonicity formula to prove the three-ball type theorem for a manifold with finitely many cylindrical ends. In the paper[3], Tobias H. Colding and William P. Minicozzi II prove monotonicity of a parabolic frequency on manifolds. In the paper [4], Tobias H. Colding and William P. Minicozzi II discuss some monotonicity formula for both parabola and elliptic operators and their geometrical meaning. In the paper [5], Tobias Holck Colding proves three monotonicity for manifolds with a lower Ricci curvature bound. In the paper[8], Jiuyi Zhu uses the monotonicity of a frequency function to prove the three-ball theorem for Schrödinger equations and higher order elliptic equations. In the paper [7], Jianyu Ou uses the same method to prove the three-ball type theorem on gradient shrinking Ricci soliton with constant scalar curvature.

Ricci solitons are generalization of Einstein metrics, special solutions to Hamilton's Ricci flow and they are important in the singularity study of the Ricci flow [1]. A complete Riemannian manifold (M, g) is called a Ricci soliton if

$$\mathrm{Ric} + \nabla\nabla f = \lambda g$$

Cigar soliton is a simple example of steady Ricci soliton which is discovered by Hamilton [6]. Its base manifold is $\mathbb{R}^2$, and its metric is

$$g = \frac{dx^2 + dy^2}{1 + x^2 + y^2},$$

By integration, we can also get the distance of the point from the vertex,

$$\rho = \ln(\sqrt{x_1^2 + x_2^2} + \sqrt{x_1^2 + x_2^2 + 1}).$$

For some eigenfunction

$$\Delta u = -\lambda u,$$

we can find the monotonicity formula on cigar soliton using similar method as Jiuyi Zhu uses in [8],

Theorem 1.1.

$$(e^{20r+7\ln r} N(r) + \lambda e^{31r})' \geq 0,$$

where $N(r) = \frac{I(r)}{H(r)} = \frac{(\alpha+1)\int_{B(r)} u<\nabla u \nabla \rho^2>(r^2-\rho^2)^\alpha dx}{\int_{B(r)} u^2(r^2-\rho^2)^\alpha dx}$ *for arbitrary* $\alpha \geq 2$.

Furthermore, by the monotonicity formula we find, we can prove the three-ball type theorem on cigar soliton

Theorem 1.2.

$$h(r_2)^{C_1+C_3} \leq (\frac{4}{3}r^{2\alpha-2})^{C_1+C_3} e^{C_4-C_2} h(r_1)^{C_1} h(r_3)^{C_3},$$

where C_1, C_2, C_3, C_4 *are constants depend only on* r_1, r_2, r_3.

We will prove Theorem 1.1. in Section 1 and Theorem 1.2. in Section 2.

2. Monotonicity formula

First, we define

$$H(r) = \int_{B(r)} u^2(r^2-\rho^2)^\alpha dx,$$

where α is any integer bigger than 1.

To find the monotonicity formula, we need to find the maximum value of $H'(r)$. To prove the three-ball type theorem, we also need the minimum value of $H'(r)$.

When $r > r_0$,

$$\begin{aligned} H'(r) \quad &= \lim_{r_0 \to r} \frac{H(r) - H(r_0)}{r - r_0} \\ &= \lim_{r_0 \to r} \frac{\int_{B(r)\setminus B(r_0)} u^2(r^2-\rho^2)^\alpha dx + \int_{B(r_0)} u^2[(r^2-\rho^2)^\alpha - (r_0^2-\rho^2)^\alpha]dx}{r - r_0} \\ &\leq \lim_{r_0 \to r} \int_{B(r)\setminus B(r_0)} u^2 \frac{(r^2-r_0^2)^\alpha}{r-r_0} dx + 2\alpha r \int_{B(r)} u^2(r^2-\rho^2)^{\alpha-1} dx \\ &= 2\alpha r \int_{B(r)} u^2(r^2-\rho^2)^{\alpha-1} dx \end{aligned}$$

On the other hand, using the same method, we can prove that $H'(r) \geq 2\alpha r \int_{B(r)} u^2(r^2-\rho^2)^{\alpha-1}dx$ when $r < r_0$.

Therefore,

$$\begin{aligned} H'(r) \quad &= 2\alpha r \int_{B(r)} u^2(r^2-\rho^2)^{\alpha-1} dx \\ &= \frac{2\alpha}{r} \int_{B(r)} u^2(r^2-\rho^2)^\alpha dx + \frac{2\alpha}{r} \int_{B(r)} u^2(r^2-\rho^2)^{\alpha-1}\rho^2 dx \\ &= \frac{2\alpha}{r} H(r) - \frac{1}{2r} \int_{B(r)} u^2 < \nabla \rho^2 \nabla (r^2-\rho^2)^\alpha > dx. \end{aligned}$$

Use integral by part for the second term in the right side

$$\begin{aligned} H'(r) &= \frac{2\alpha}{r}H(r) + \frac{1}{r}\int_{B(r)} u < \nabla u \nabla \rho^2 > (r^2-\rho^2)^\alpha dx \\ &\quad + \frac{1}{2r}\int_{B(r)} u^2 \Delta \rho^2 (r^2-\rho^2)^\alpha dx \\ &= \frac{2\alpha+1}{r}H(r) + \frac{1}{r}\int_{B(r)} u < \nabla u \nabla \rho^2 > (r^2-\rho^2)^\alpha dx \\ &\quad + \frac{1}{r}\int_{B(r)} u^2 \frac{\rho}{\sqrt{x_1^2+x_2^2}\sqrt{x_1^2+x_2^2+1}}(r^2-\rho^2)^\alpha dx. \end{aligned}$$

The second term in the right side would later be used in the frequency function. Therefore, we only need to find the minimum and maximum values of the third term in the right hand side.

$$\begin{aligned} 0 &\le \frac{1}{r}\int_{B(r)} u^2 \frac{\rho}{\sqrt{x_1^2+x_2^2}\sqrt{x_1^2+x_2^2+1}}(r^2-\rho^2)^\alpha dx \\ &= \frac{1}{r}\int_{B(r)} u^2 \frac{4\rho}{e^{2\rho}-e^{-2\rho}}(r^2-\rho^2)^\alpha dx \\ &\le \frac{1}{r}\int_{B(r)} u^2 \frac{2}{e^{2\rho_0}+e^{-2\rho_0}}(r^2-\rho^2)^\alpha dx \\ &\le \frac{1}{r}H(r), \end{aligned}$$

where ρ_0 is one of the solutions of $(\frac{4\rho}{e^{2\rho}-e^{-2\rho}})' = 0$ and where $\frac{4\rho}{e^{2\rho}-e^{-2\rho}}$ has its maximum value.

So we find the minimum and maximum values of $H'(r)$ now

$$\begin{aligned} &\frac{2\alpha+1}{r}H(r) + \frac{1}{r}\int_{B(r)} u < \nabla u \nabla \rho^2 > (r^2-\rho^2)^\alpha dx \\ &\le H'(r) \le \frac{2\alpha+2}{r}H(r) + \frac{1}{r}\int_{B(r)} u < \nabla u \nabla \rho^2 > (r^2-\rho^2)^\alpha dx. \end{aligned}$$

Let

$$I(r) = (\alpha+1)\int_{B(r)} u < \nabla u \nabla \rho^2 > (r^2-\rho^2)^\alpha dx,$$

and we get

$$\frac{2\alpha+1}{r}H(r) + \frac{1}{(\alpha+1)r}I(r) \le H'(r) \le \frac{2\alpha+2}{r}H(r) + \frac{1}{(\alpha+1)r}I(r). \tag{2.1}$$

Also, we let $N(r) = \frac{I(r)}{H(r)}$. Later we will try to prove the monotonicity of the transformation of $N(r)$.

Integrate by part to transform $I(r)$

$$
\begin{aligned}
I(r) \quad &= (\alpha+1)\int_{B(r)} u < \nabla u \nabla \rho^2 > (r^2-\rho^2)^{\alpha} dx \\
&= \int_{B(r)} |\nabla u|^2 (r^2-\rho^2)^{\alpha+1} dx + \int_{B(r)} u\Delta u (r^2-\rho^2)^{\alpha+1} dx \\
&= \int_{B(r)} |\nabla u|^2 (r^2-\rho^2)^{\alpha+1} dx - \lambda \int_{B(r)} u^2 (r^2-\rho^2)^{\alpha+1} dx.
\end{aligned}
$$

To find the monotonicity formula, we need to find the minimum value of $I'(r)$

$$
\begin{aligned}
I'(r) \quad &= 2(\alpha+1)r \int_{B(r)} |\nabla u|^2 (r^2-\rho^2)^{\alpha} dx \\
&\quad -2(\alpha+1)r\lambda \int_{B(r)} u^2 (r^2-\rho^2)^{\alpha} dx \\
&= \frac{2(\alpha+1)}{r} \int_{B(r)} |\nabla u|^2 (r^2-\rho^2)^{\alpha+1} dx \\
&\quad + \frac{2(\alpha+1)}{r} \int_{B(r)} |\nabla u|^2 (r^2-\rho^2)^{\alpha} \rho^2 dx \\
&\quad -2(\alpha+1)r\lambda \int_{B(r)} u^2 (r^2-\rho^2)^{\alpha} dx.
\end{aligned}
$$

Since $|\nabla \rho|^2 = 1$, we can add it to the second term in the right hand side

$$
\begin{aligned}
I'(r) \quad &= \frac{2(\alpha+1)}{r} \int_{B(r)} |\nabla u|^2 (r^2-\rho^2)^{\alpha+1} dx \\
&\quad + \frac{2(\alpha+1)}{r} \int_{B(r)} \rho^2 |\nabla \rho|^2 |\nabla u|^2 (r^2-\rho^2)^{\alpha} dx \\
&\quad -2(\alpha+1)r\lambda \int_{B(r)} u^2 (r^2-\rho^2)^{\alpha} dx \\
&= \frac{2(\alpha+1)}{r} \int_{B(r)} |\nabla u|^2 (r^2-\rho^2)^{\alpha+1} dx \\
&\quad - \frac{1}{2r} \int_{B(r)} |\nabla u|^2 < \nabla \rho^2 \nabla (r^2-\rho^2)^{\alpha+1} > dx \\
&\quad -2(\alpha+1)r\lambda \int_{B(r)} u^2 (r^2-\rho^2)^{\alpha} dx.
\end{aligned}
$$

Use integral by part for the second term in the right hand side, we get

$$\begin{aligned} I'(r) \quad &= \frac{2(\alpha+1)}{r}\int_{B(r)} |\nabla u|^2 (r^2-\rho^2)^{\alpha+1}dx \\ &+\frac{1}{2r}\int_{B(r)} <\nabla|\nabla u|^2\nabla\rho^2> (r^2-\rho^2)^{\alpha+1}dx \\ &+\frac{1}{2r}\int_{B(r)} |\nabla u|^2 (r^2-\rho^2)^{\alpha+1}\Delta\rho^2 dx \\ &-2(\alpha+1)r\lambda\int_{B(r)} u^2(r^2-\rho^2)^{\alpha}dx, \end{aligned}$$

Since $\Delta\rho^2 = 2\rho\Delta\rho + 2|\nabla\rho|^2 \geq 2$

$$\begin{aligned} I'(r) \quad &\geq \frac{2\alpha+3}{r}\int_{B(r)} |\nabla u|^2 (r^2-\rho^2)^{\alpha+1}dx \\ &+\frac{1}{2r}\int_{B(r)} <\nabla|\nabla u|^2\nabla\rho^2> (r^2-\rho^2)^{\alpha+1}dx \\ &-2(\alpha+1)r\lambda\int_{B(r)} u^2(r^2-\rho^2)^{\alpha}dx. \end{aligned}$$

Now we divide the second term in the right side to find its minimum value

$$\begin{aligned} &\frac{1}{2r}\int_{B(r)} <\nabla|\nabla u|^2\nabla\rho^2> (r^2-\rho^2)^{\alpha+1}dx \\ &=\frac{1}{2r}\int_{B(r)} (1+x_1^2+x_2^2)\partial_j[(1+x_1^2+x_2^2)\partial_i u\partial_i u]\partial_j\rho^2(r^2-\rho^2)^{\alpha+1}dx \\ &=\frac{1}{2r}\int_{B(r)} (1+x_1^2+x_2^2)\partial_j(1+x_1^2+x_2^2)\partial_i u\partial_i u\partial_j\rho^2(r^2-\rho^2)^{\alpha+1}dx \\ &\quad+\frac{1}{r}\int_{B(r)} (1+x_1^2+x_2^2)^2\partial_i u\partial_{ij}u\partial_j\rho^2(r^2-\rho^2)^{\alpha+1}dx. \end{aligned}$$

Because

$$(1+x_1^2+x_2^2)\geq 0,\ \partial_i u\partial_i u\geq 0,\ \partial_j(1+x_1^2+x_2^2)\partial_j\rho^2\geq 0,\ (r^2-\rho^2)^{\alpha+1}\geq 0.$$

Therefore,

$$\frac{1}{2r}\int_{B(r)} (1+x_1^2+x_2^2)\partial_j(1+x_1^2+x_2^2)\partial_i u\partial_i u\partial_j\rho^2(r^2-\rho^2)^{\alpha+1}dx \geq 0,$$

and

$$\begin{aligned} &\frac{1}{2r}\int_{B(r)} <\nabla|\nabla u|^2\nabla\rho^2> (r^2-\rho^2)^{\alpha+1}dx \\ &\geq \frac{1}{r}\int_{B(r)} (1+x_1^2+x_2^2)^2\partial_i u\partial_{ij}u\partial_j\rho^2(r^2-\rho^2)^{\alpha+1}dx. \end{aligned}$$

Use integral by part, we get

$$
\begin{aligned}
&\frac{1}{2r}\int_{B(r)} <\nabla|\nabla u|^2\nabla\rho^2>(r^2-\rho^2)^{\alpha+1}dx\\
&\geq \frac{\lambda}{r}\int_{B(r)} u<\nabla u\nabla\rho^2>(r^2-\rho^2)^{\alpha+1}dx\\
&\quad -\frac{1}{r}\int_{B(r)}(1+x_1^2+x_2^2)^2\partial_j u\partial_i u\partial_{ij}\rho^2(r^2-\rho^2)^{\alpha+1}dx\\
&\quad +\frac{\alpha+1}{r}\int_{B(r)}<\nabla u\nabla\rho^2>^2(r^2-\rho^2)^{\alpha}dx\\
&\quad -\frac{2}{r}\int_{B(r)}<\nabla u\nabla\rho^2>\partial_i u\partial_i(1+x_1^2+x_2^2)(r^2-\rho^2)^{\alpha+1}dx\\
&= -\frac{\lambda}{2r}\int_{B(r)} u^2\Delta\rho^2(r^2-\rho^2)^{\alpha+1}dx\\
&\quad +\frac{2(\alpha+1)\lambda}{r}\int_{B(r)} u^2\rho^2(r^2-\rho^2)^{\alpha}dx\\
&\quad -\frac{1}{r}\int_{B(r)}(1+x_1^2+x_2^2)^2\partial_j u\partial_i u\partial_{ij}\rho^2(r^2-\rho^2)^{\alpha+1}dx\\
&\quad +\frac{\alpha+1}{r}\int_{B(r)}<\nabla u\nabla\rho^2>^2(r^2-\rho^2)^{\alpha}dx\\
&\quad -\frac{2}{r}\int_{B(r)}<\nabla u\nabla\rho^2>\partial_i u\partial_i(1+x_1^2+x_2^2)(r^2-\rho^2)^{\alpha+1}dx.
\end{aligned}
$$

Put it back into $I'(r)$, and we get

$$
\begin{aligned}
I'(r) \quad &\geq \frac{2(\alpha+1)}{r}I(r)+\frac{1}{r}\int_{B(r)}|\nabla u|^2(r^2-\rho^2)^{\alpha+1}dx\\
&\quad -\frac{\lambda}{2r}\int_{B(r)} u^2\Delta\rho^2(r^2-\rho^2)^{\alpha+1}dx\\
&\quad -\frac{1}{r}\int_{B(r)}(1+x_1^2+x_2^2)^2\partial_j u\partial_i u\partial_{ij}\rho^2(r^2-\rho^2)^{\alpha+1}dx\\
&\quad +\frac{\alpha+1}{r}\int_{B(r)}<\nabla u\nabla\rho^2>^2(r^2-\rho^2)^{\alpha}dx\\
&\quad -\frac{2}{r}\int_{B(r)}<\nabla u\nabla\rho^2>\partial_i u\partial_i(1+x_1^2+x_2^2)(r^2-\rho^2)^{\alpha+1}dx.
\end{aligned}
$$

Since the third term in the right side has a negative coefficient, we need to find its maximum value

$$
\begin{aligned}
\frac{\lambda}{2r}\int_{B(r)} u^2\Delta\rho^2(r^2-\rho^2)^{\alpha+1}dx \quad &\leq \frac{2\lambda}{r}\int_{B(r)} u^2(r^2-\rho^2)^{\alpha+1}\\
&\leq 2r\lambda\int_{B(r)} u^2(r^2-\rho^2)^{\alpha}.
\end{aligned}
$$

Also, we need to find the maximum value for the fourth term in the right side for its negative coefficient.

$$\begin{aligned}
&\frac{1}{r}\int_{B(r)}(1+x_1^2+x_2^2)^2\partial_j u\partial_i u\partial_{ij}\rho^2(r^2-\rho^2)^{\alpha+1}dx\\
&=\frac{1}{r}\int_{B(r)}(1+x_1^2+x_2^2)^2\partial_1 u\partial_1 u\partial_{11}\rho^2(r^2-\rho^2)^{\alpha+1}dx\\
&\quad+\frac{1}{r}\int_{B(r)}(1+x_1^2+x_2^2)^2\partial_2 u\partial_2 u\partial_{22}\rho^2(r^2-\rho^2)^{\alpha+1}dx\\
&\quad+\frac{2}{r}\int_{B(r)}(1+x_1^2+x_2^2)^2\partial_1 u\partial_2 u\partial_{12}\rho^2(r^2-\rho^2)^{\alpha+1}dx.
\end{aligned}$$

$$\begin{aligned}
\partial_{11}\rho^2 \quad &=2(\partial_1\rho)^2+2\rho\partial_{11}\rho\\
&=\frac{2x_1^2}{(x_1^2+x_2^2)(1+x_1^2+x_2^2)}\\
&\quad+2\ln(\sqrt{x_1^2+x_2^2}+\sqrt{1+x_1^2+x_2^2})\frac{-x_1^4+x_2^4+x_2^2}{(x_1^2+x_2^2)^{\frac{3}{2}}(1+x_1^2+x_2^2)^{\frac{3}{2}}}\\
&\leq\frac{2r+4}{1+x_1^2+x_2^2}.
\end{aligned}$$

We can also prove that

$$\partial_{22}\rho^2\leq\frac{2r+4}{1+x_1^2+x_2^2},\ \partial_{12}\rho^2\leq\frac{2r+4}{1+x_1^2+x_2^2},$$

using the same method

Therefore,

$$\begin{aligned}
&\frac{1}{r}\int_{B(r)}(1+x_1^2+x_2^2)^2\partial_j u\partial_i u\partial_{ij}\rho^2(r^2-\rho^2)^{\alpha+1}dx\\
&\leq\frac{4r+8}{r}\int_{B(r)}|\nabla u|^2(r^2-\rho^2)^{\alpha+1}dx.
\end{aligned}$$

Now to find the maximum value for the last term in the right side

$$\begin{aligned}
&\frac{2}{r}\int_{B(r)} < \nabla u \nabla \rho^2 > \partial_i u \partial_i (1+x_1^2+x_2^2)(r^2-\rho^2)^{\alpha+1}dx \\
&= \frac{1}{r}\int_{B(r)} \partial_j u \partial_j \rho^2 \partial_i u \partial_i (1+x_1^2+x_2^2)^2 (r^2-\rho^2)^{\alpha+1}dx \\
&= \frac{1}{r}\int_{B(r)} \partial_1 u \partial_1 \rho^2 \partial_1 u \partial_1 (1+x_1^2+x_2^2)^2 (r^2-\rho^2)^{\alpha+1}dx \\
&\quad +\frac{1}{r}\int_{B(r)} \partial_2 u \partial_2 \rho^2 \partial_2 u \partial_2 (1+x_1^2+x_2^2)^2 (r^2-\rho^2)^{\alpha+1}dx \\
&\quad +\frac{1}{r}\int_{B(r)} \partial_1 u \partial_1 \rho^2 \partial_2 u \partial_2 (1+x_1^2+x_2^2)^2 (r^2-\rho^2)^{\alpha+1}dx \\
&\quad +\frac{1}{r}\int_{B(r)} \partial_2 u \partial_2 \rho^2 \partial_1 u \partial_1 (1+x_1^2+x_2^2)^2 (r^2-\rho^2)^{\alpha+1}dx.
\end{aligned}$$

$$\begin{aligned}
\partial_1 \rho^2 \partial_1 (1+x_1^2+x_2^2)^2 \qquad &= 2\rho \frac{x_1}{\sqrt{x_1^2+x_2^2}\sqrt{1+x_1^2+x_2^2}} 2(1+x_1^2+x_2^2)2x_1 \\
&\le 8r(1+x_1^2+x_2^2).
\end{aligned}$$

Similarly, we can prove that

$$\partial_2 \rho^2 \partial_2 (1+x_1^2+x_2^2)^2 \le 8r(1+x_1^2+x_2^2),\ \partial_1 \rho^2 \partial_2 (1+x_1^2+x_2^2)^2 \le 8r(1+x_1^2+x_2^2),$$

and

$$\partial_2 \rho^2 \partial_1 (1+x_1^2+x_2^2)^2 \le 8r(1+x_1^2+x_2^2).$$

using the same method.

Therefore,

$$\begin{aligned}
&\frac{2}{r}\int_{B(r)} < \nabla u \nabla \rho^2 > (r^2-\rho^2)^{\alpha+1} \partial_i u \partial_i (1+x_1^2+x_2^2) dx \\
&\le 16 \int_{B(r)} |\nabla u|^2 (r^2-\rho^2)^{\alpha+1}dx.
\end{aligned}$$

Put them back to $I'(r)$, we obtain

$$\begin{aligned}
I'(r) \quad &\geq \frac{2(\alpha+1)}{r}I(r)+\frac{1}{r}\int_{B(r)}|\nabla u|^2(r^2-\rho^2)^{\alpha+1}dx \\
&-2r\lambda\int_{B(r)}u^2(r^2-\rho^2)^{\alpha}dx \\
&-\frac{20r+8}{r}\int_{B(r)}|\nabla u|^2(r^2-\rho^2)^{\alpha+1}dx \\
&+\frac{\alpha+1}{r}\int_{B(r)}<\nabla u\nabla\rho^2>^2(r^2-\rho^2)^{\alpha}dx \\
&\geq \frac{2(\alpha+1)}{r}I(r)-2r\lambda\int_{B(r)}u^2(r^2-\rho^2)^{\alpha}dx-\frac{20r+7}{r}I(r)dx \\
&-\frac{(20r+7)\lambda}{r}\int_{B(r)}u^2(r^2-\rho^2)^{\alpha+1}dx \\
&+\frac{\alpha+1}{r}\int_{B(r)}<\nabla u\nabla\rho^2>^2(r^2-\rho^2)^{\alpha}dx \\
&\geq \frac{2(\alpha+1)}{r}I(r)-\frac{20r+7}{r}I(r) \\
&-(20r^2+9r)\lambda H(r)+\frac{\alpha+1}{r}\int_{B(r)}<\nabla u\nabla\rho^2>^2(r^2-\rho^2)^{\alpha}dx.
\end{aligned}$$

To find the monotonicity of $N(r)$ or its transformation, we need to prove that its derivative is always positive.

$$\begin{aligned}
N'(r) \quad &= \frac{H(r)I'(r)-I(r)H'(r)}{H^2(r)} \\
&\geq \frac{1}{H^2(r)}\{H(r)[\frac{2(\alpha+1)}{r}I(r)-\frac{20r+7}{r}I(r)-(20r^2+9r)\lambda H(r) \\
&+\frac{\alpha+1}{r}\int_{B(r)}<\nabla u\nabla\rho^2>^2(r^2-\rho^2)^{\alpha}dx] \\
&-I(r)[\frac{2(\alpha+1)}{r}H(r)+\frac{1}{(\alpha+1)r}I(r)]\} \\
&\geq \frac{1}{H^2(r)}\{H(r)[-\frac{20r+7}{r}I(r)-(20r^2+9r)\lambda H(r) \\
&+\frac{\alpha+1}{r}\int_{B(r)}<\nabla u\nabla\rho^2>^2(r^2-\rho^2)^{\alpha}dx]-\frac{1}{(\alpha+1)r}I^2(r)\}.
\end{aligned}$$

According to Cauchy-Schwarz inequality,

$$\begin{aligned}
&\int_{B(r)}<\nabla u\nabla\rho^2>^2(r^2-\rho^2)^{\alpha}dx\int_{B(r)}u^2(r^2-\rho^2)^{\alpha}dx \\
&\geq (\int_{B(r)}u<\nabla u\nabla\rho^2>(r^2-\rho^2)^{\alpha}dx)^2.
\end{aligned}$$

Thus,

$$\begin{aligned} N'(r) \quad &\geq \frac{1}{H^2(r)}\{H(r)[-\frac{20r+7}{r}I(r)-(20r^2+9r)\lambda H(r)]\} \\ &\geq -\frac{20r+7}{r}N(r)-(20r^2+9r)\lambda. \end{aligned}$$

That means

$$\begin{aligned} 0 \quad &\leq rN'(r)+(20r+7)N(r)+(20r^3+9r^2)\lambda \\ &= \frac{1}{e^{20r+7\ln r}}[(e^{20r+7\ln r}N(r))'+(20r^3+9r^2)\lambda e^{20r+7\ln r}] \\ &\leq \frac{1}{e^{20r+7\ln r}}[(e^{20r+7\ln r}N(r))'+29\lambda e^{30r}] \\ &\leq \frac{1}{e^{20r+7\ln r}}(e^{20r+7\ln r}N(r)+\lambda e^{31r})'. \end{aligned}$$

Therefore,

$$(e^{20r+7\ln r}N(r)+\lambda e^{31r})' \geq 0.$$

Theorem 1.1 is proved.

3. Three-ball type theorem

Here, by using the monotonicity formula we find in section 2, we aim to prove the three-ball type theorem on cigar soliton.

By (**??**)

$$\frac{H'(r)}{H(r)} \leq \frac{2\alpha+2}{r}+\frac{1}{(\alpha+1)r}N(r),$$

$$(\ln H(r))' \leq (2\alpha+2)(\ln r)'+\frac{1}{(\alpha+1)r}N(r).$$

That means integral from r_1 to $2r_2$

$$\int_{r_1}^{2r_2}(\ln H(r))'dr \leq \int_{r_1}^{2r_2}(2\alpha+2)(\ln r)'dr+\int_{r_1}^{2r_2}\frac{1}{(\alpha+1)r}N(r)dr.$$

So

$$\ln\frac{H(2r_2)}{H(r_1)} \quad \leq (2\alpha+2)\ln\frac{2r_2}{r_1}+\frac{1}{\alpha+1}\int_{r_1}^{2r_2}\frac{1}{r}N(r)dr.$$

Now we use the monotonicity formula

$$\begin{aligned}\ln\frac{H(2r_2)}{H(r_1)} &\leq (2\alpha+2)\ln\frac{2r_2}{r_1}+\frac{1}{\alpha+1}\int_{r_1}^{2r_2}\frac{e^{20r+7\ln r}N(r)+\lambda e^{31r}}{e^{20r+7\ln r}r}\\ &-\frac{\lambda e^{31r}}{e^{20r+7\ln r}r}dr\\ &\leq (2\alpha+2)\ln\frac{2r_2}{r_1}\\ &+\frac{e^{40r_2+7\ln 2r_2}N(2r_2)+\lambda e^{62r_2}}{\alpha+1}\int_{r_1}^{2r_2}\frac{1}{e^{20r+7\ln r}r}dr\\ &-\frac{\lambda e^{11r_1}}{256(\alpha+1)r_2^8}(2r_2-r_1).\end{aligned}$$

Similarly, because

$$\frac{H'(r)}{H(r)}\geq\frac{2\alpha+1}{r}+\frac{1}{(\alpha+1)r}N(r),$$

we can derive that

$$\begin{aligned}\ln\frac{H(r_3)}{H(2r_2)} &\geq (2\alpha+1)\ln\frac{r_3}{2r_2}\\ &+\frac{e^{40r_2+7\ln 2r_2}N(2r_2)+\lambda e^{62r_2}}{\alpha+1}\int_{2r_2}^{r_3}\frac{1}{e^{20r+7\ln r}r}dr\\ &-\frac{\lambda e^{11r_3}}{256(\alpha+1)r_2^8}(r_3-2r_2).\end{aligned}$$

Therefore,

$$\begin{aligned}N(2r_2) &\geq \frac{(\alpha+1)r_1^{28}}{128e^{40r_2}r_2^7(2r_2-r_1)}\ln\frac{H(2r_2)}{H(r_1)}-\frac{(\alpha+1)^2e^{16r_2}\ln\frac{2r_2}{r_1}}{64r_2^7(2r_2-r_1)}\\ &-\frac{\lambda e^{22r_2}}{128r_2^7}+\frac{\lambda e^{11r_1}r_1^{28}}{2^{15}e^{40r_2}r_2^{15}}\\ &\geq \frac{(\alpha+1)r_1^{28}}{128e^{40r_2}r_2^7(2r_2-r_1)}\ln\frac{H(2r_2)}{H(r_1)}-\frac{(\alpha+1)^2e^{16r_2}\ln\frac{2r_2}{r_1}}{64r_2^7(2r_2-r_1)}\\ &-\frac{\lambda e^{22r_2}}{128r_2^7}.\end{aligned}$$

On the other hand,

$$\begin{aligned}N(2r_2) &\leq \frac{(\alpha+1)e^{28r_3}}{128e^{40r_2}r_2^7(r_3-2r_2)}\ln\frac{H(r_3)}{H(2r_2)}-\frac{2^{21}(2\alpha^2+3\alpha+1)r_2^{21}\ln\frac{r_3}{2r_2}}{e^{40r_2}(r_3-2r_2)}\\ &-\frac{\lambda e^{22r_2}}{128r_2^7}+\frac{\lambda e^{39r_3}}{2^{15}e^{40r_2}r_2^{15}}\\ &\leq \frac{(\alpha+1)e^{28r_3}}{128e^{40r_2}r_2^7(r_3-2r_2)}\ln\frac{H(r_3)}{H(2r_2)}+\frac{\lambda e^{39r_3}}{2^{15}e^{40r_2}r_2^{15}}.\end{aligned}$$

Let $C_1 = \frac{(\alpha+1)r_1^{28}}{128e^{40r_2}r_2^7(2r_2-r_1)}$, $C_2 = -\frac{(\alpha+1)^2e^{16r_2}\ln\frac{2r_2}{r_1}}{64r_2^7(2r_2-r_1)} - \frac{\lambda e^{22r_2}}{128r_2^7}$,
$C_3 = \frac{(\alpha+1)e^{28r_3}}{128e^{40r_2}r_2^7(r_3-2r_2)}$, $C_4 = \frac{\lambda e^{39r_3}}{2^{15}e^{40r_2}r_2^{15}}$.

Then,

$$C_1 \ln\frac{H(2r_2)}{H(r_1)} + C_2 \le C_3 \ln\frac{H(r_3)}{H(2r_2)} + C_4,$$

$$(C_1 + C_3)\ln H(2r_2) \le C_1 \ln H(r_1) + C_3 \ln H(r_3) + (C_4 - C_2).$$

Here we define

$$h(r) = \int_{B(r)} u^2 dx.$$

It's clear that:

$$H(r) \le \int_{B(r)} u^2 r^{2\alpha} dx = r^{2\alpha} h(r).$$

Let $0 < t < r$,

$$h(t) = \int_{B(t)} u^2 dx \le \int_{B(t)} u^2 (\frac{r^2-\rho^2}{r^2-t^2})^\alpha dx = \frac{H(r)}{(r^2-t^2)^\alpha} \le \frac{H(r)}{r^2-t^2}.$$

Let $t = \frac{r}{2}$, then

$$\frac{3}{4} r^2 h(\frac{r}{2}) \le H(r) \le r^{2\alpha} h(r).$$

Thus,

$$\begin{aligned}(C_1 + C_3)\ln\frac{3}{4}r^2 + (C_1 + C_3)\ln h(r_2) \quad &\le (C_1 + C_3)\ln r^{2\alpha} + C_1 \ln h(r_1) \\ &\quad + C_3 \ln h(r_3) + (C_4 - C_2).\end{aligned}$$

We obtain

$$(C_1 + C_3)\ln h(r_2) \le C_1 \ln h(r_1) + C_3 \ln h(r_3) + (C_1 + C_3)\ln\frac{4}{3}r^{2\alpha-2} + (C_4 - C_2),$$

which implies

$$h(r_2)^{C_1+C_3} \le (\frac{4}{3}r^{2\alpha-2})^{C_1+C_3} e^{C_4-C_2} h(r_1)^{C_1} h(r_3)^{C_3},$$

where $C_1 = \frac{(\alpha+1)r_1^{28}}{128e^{40r_2}r_2^7(2r_2-r_1)}$, $C_2 = -\frac{(\alpha+1)^2e^{16r_2}\ln\frac{2r_2}{r_1}}{64r_2^7(2r_2-r_1)} - \frac{\lambda e^{22r_2}}{128r_2^7}$, $C_3 = \frac{(\alpha+1)e^{28r_3}}{128e^{40r_2}r_2^7(r_3-2r_2)}$, $C_4 = \frac{\lambda e^{39r_3}}{2^{15}e^{40r_2}r_2^{15}}$.

Thus, Theorem 1.2 is proved.

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