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Types of graphs generated by Pascal's simplices and a problem proposed by Teguia and Godbole

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Abstract

It is known that the divisibility of numbers in Pascal's simplices possesses fractal properties. In this paper, we consider the graphs $A_t(m,n)$ generated by Pascal's t-simplex modulo m. We prove that these graphs generalize the Sierpiński graphs introduced by Klavžar and Milutinović. We study their properties such as diameter and Hamiltonicity, and apply our result to solve an open problem proposed by Teguia and Godbole. We also give lower and upper bounds of the optimal pebbling number π_{opt} of triangular grid graphs.

Keywords: Pascal's triangle; Pascal's simplex; triangular grid; Sierpiński gasket graph; Hamiltonicity; optimal pebbling number.

Contents

1 Introduction

Pascal's triangle, also known as the arithmetic triangle, is one of the most well-known mathematical objects. A triangular array of binomial coefficients, it is a favorite with mathematical expositors [2] [17], and has many real-world applications. For example, it has been used to model the Tower of Hanoi game [9], design a cryptosystem [5], and solve PDEs [16]. It has even proved to be useful in physics [1] [10].

One of the most striking properties of Pascal's triangle is that the odd numbers in it form a pattern similar to the Sierpiński gasket, a fractal object obtained from an equilateral triangle by removing smaller triangles from it recursively.

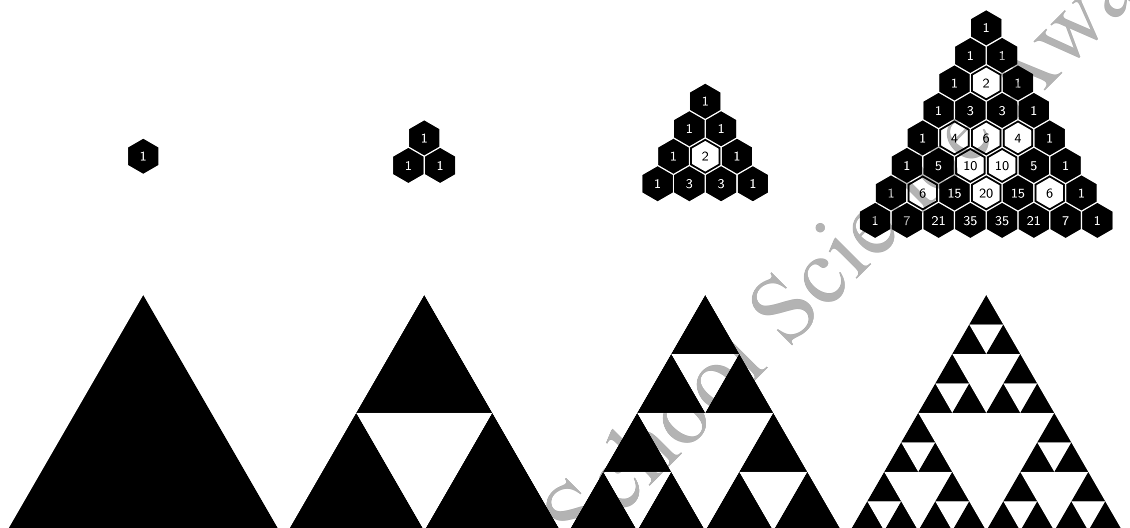


Figure 1: Pascal's triangle modulo 2 approximates the Sierpinski gasket.

Other patterns arise when the modulo m is not 2. When m is a prime, the pattern is similar to the case where $m = 2$. When m is a prime power, the pattern becomes more complicated. And when m has multiple different prime divisors, the pattern is more complicated still, as it is the union of the patterns generated by the prime powers that divide m. These patterns were considered by Holter et al. [10], who called them the Pascal-Sierpiński gaskets.

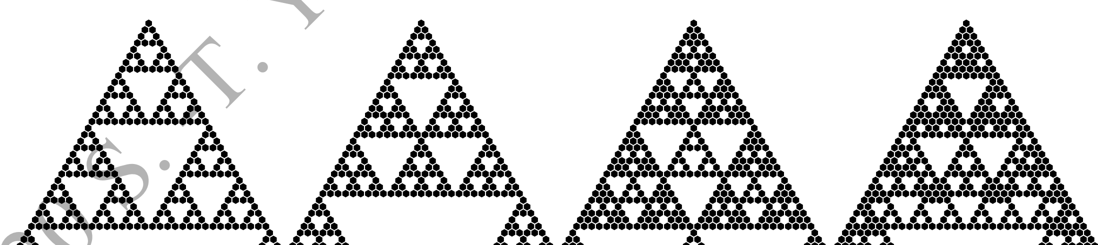


Figure 2: Numbers not divisible by m in Pascal's triangle[1], $m = 2, 3, 4, 6$.

[1] I have made an interactive applet available at `https://www.findegil.xyz/applets/pascal` that shows these patterns.

It is natural to consider the generalizations of the arithmetic triangle into the t-dimensional Euclidean space. For example, the 3-dimensional generalization is called Pascal's pyramid [19], in which trinomial coefficients are placed on a tetrahedral lattice.

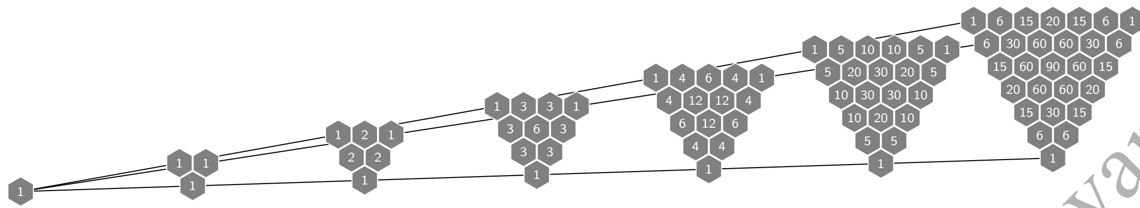


Figure 3: Pascal's pyramid.

In geometry, the t-dimensional generalization of the triangle is called the t-simplex. For example, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron. Similarly, the t-dimensional generalization of Pascal's triangle is called Pascal's simplex [22]. It is rather unsurprising that the numbers not divisible by m in Pascal's simplex also yield fractal-like patterns [18].

In this paper, we consider the graphs $A_t(m,n)$ generated by Pascal's t-simplex modulo m. The vertices of $A_t(m,n)$ are the multinomial coefficients in the first $n+1$ components (when $t=2$, a component is a row) of Pascal's t-simplex that are not divisible by m, and edges are drawn between geometrically adjacent vertices. For prime p and positive integer k, $A_{t_p^k}$ denotes $A_t(p,p^k-1)$. In Sections 2 to 4, we present a rigorous definition of the graphs $A_t(m,n)$, and discuss some of their properties, such as symmetry, self-similarity and the number of their vertices and edges.

Hinz [9] considered the graphs $A_{2_2^k}$, and proved that it is isomorphic to TH_n, the Tower of Hanoi graph with n pegs. The vertices of TH_n are the legal states of the discs in the Tower of Hanoi, with two vertices adjacent if one can be obtained from another with a legal move. Klavžar and Milutinović [14] proposed the Sierpiński graphs S_n^k, which were motivated by a topological construction called the Lipscomb space [15], and proved that S_3^k is isomorphic to TH_k (and therefore isomorphic to $A_{2_2^k}$). In Section 5, we prove the more general statement that the graph $A_{t_2^k}$ is isomorphic to the Sierpiński graph S_{t+1}^k. In Sections 6 and 7, we investigate the diameters and Hamiltonicity of $A_t(m,n)$. Teguia and Godbole [20] studied the Sierpiński gasket graphs S_n (not to be confused with the Sierpiński graphs S_n^k), which are generated by the iterative process shown in Figure 1 that defines the Sierpiński gasket. They also proposed some open problems. One such problem concerns the properties of a generalization of the Sierpiński gasket graph generated by the numbers not divisible by p in Pascal's triangle, where p is a prime other than 2. In Section 8, we make use of our previous results to answer this problem. Another of their open questions asks about the pebbling number of S_n. We have not been able to solve this problem, but we have made some progress. In Section 9, we give lower and upper bounds of the optimal pebbling number π_{opt} (the definition of which will be presented in the section) of triangular grid graphs. In the appendix, we present an algorithm to find Hamiltonian cycles of graphs $A_{t_p^k}$.

2 Definitions and basic properties

In Pascal's simplex, the multinomial coefficients are placed on a simplex grid. Therefore, it is convenient to first introduce the simplex grid graphs, a natural generalization of the triangular grid graphs [21], and then define the graphs $A_t(m,n)$ as their induced subgraphs.

Definition 2.1. The t-simplex grid graph is denoted by $G_t(n)$. We assume that $t \geq 2$ and $n \geq 1$ unless otherwise stated. Its vertices are ordered $(t+1)$-tuples of nonnegative integers $(x_0, x_1, \ldots, x_t)$ that sum to n, and two vertices $(x_0, x_1, \dots, x_t)$ and $(y_0, y_1, \dots, y_t)$ are adjacent if $\frac{1}{2}\sum_{i=0}^{t}|x_i - y_i| = 1$.

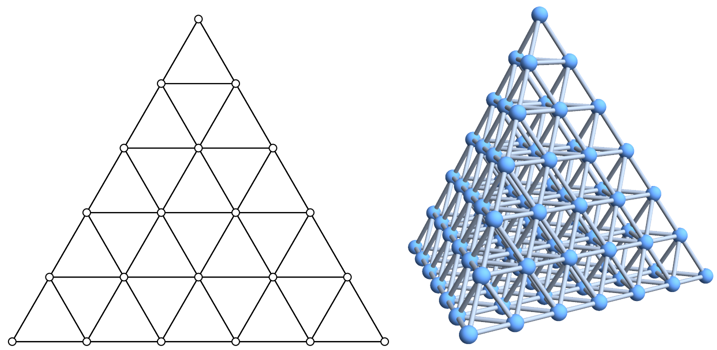


Figure 4: Simplex grid graphs $G_2(5)$ and $G_3(6)$.

Remark. Note that $\sum_{i=0}^{t}|x_i - y_i| \equiv \sum_{i=0}^{t}(x_i - y_i) = 0 \pmod 2$, so $\frac{1}{2}\sum_{i=0}^{t}|x_i - y_i|$ is always a nonnegative integer, and therefore 1 is its least possible nonzero value.

Proposition 2.1. *The condition $\frac{1}{2}\sum_{i=0}^{t}|x_i - y_i| = 1$ holds if and only if there exist integers $a, b \in \{0, 1, \dots, t\}$, $a \neq b$ such that $x_a = y_a + 1$, $y_b = x_b + 1$, and $x_c = y_c$ for $c \in \{0, 1, \dots, t\} - \{a, b\}$.*

Proof. First observe that for $a \in \{0, 1, \dots, t\}$,

$$\begin{aligned} |x_a - y_a| &= \frac{1}{2}\left(|x_a - y_a| + |(n - x_a) - (n - y_a)|\right) \\ &= \frac{1}{2}\left(|x_a - y_a| + \left|\sum_{i \in \{0,1,\dots,t\}-\{a\}} x_i - \sum_{i \in \{0,1,\dots,t\}-\{a\}} y_i\right|\right) \\ &\leq \frac{1}{2}\sum_{i=0}^{t}|x_i - y_i| = 1. \end{aligned}$$

Therefore, there exist integers $a, b \in \{0, 1, \dots, t\}$, $a \neq b$ such that $|x_a - y_a| = |x_b - y_b| = 1$, and $x_c = y_c$ for $c \in \{0, 1, \dots, t\} - \{a, b\}$. Now observe that $(x_a - y_a) + (x_b - y_b) = \sum_{i=0}^{t}(x_i - y_i) = n - n = 0$, so either $x_a = y_a + 1$, $y_b = x_b + 1$ or $x_b = y_b + 1$, $y_a = x_a + 1$. Either way, the proposition holds. ☐

Recall the definition of the sign function that $\operatorname{sgn}(x) = 0$ when $x = 0$ and $\operatorname{sgn}(x) = 1$ when $x > 0$. In the following proposition, we represent the degree of a vertex in $G_t(n)$ using the sign function. Note that such a representation may seem complicated but will make it easier for us to find the number of edges in $G_t(n)$ in the proof of Proposition 4.1.

Proposition 2.2. *The degree of vertex $(x_0, x_1, \dots, x_t)$ in $G_t(n)$ is*

$$\deg((x_0, x_1, \dots, x_t)) = t\sum_{i=0}^{t}\operatorname{sgn}(x_i).$$

Proof. The neighbors of $(x_0, x_1, \dots, x_t)$ are the ordered tuples $(y_0, y_1, \dots, y_t)$ of integers satisfying two conditions: 1) y_i is nonnegative for $0 \leq i \leq t$ and 2) there exists $a, b \in \{0, 1, \dots, t\}$, $a \neq b$ such that $x_a = y_a + 1$, $y_b = x_b + 1$, and $x_c = y_c$ for $c \in \{0, 1, \dots, t\} - \{a, b\}$. Now for every $a, b \in \{0, 1, \dots, t\}$, $a \neq b$, there is exactly one such tuple if $x_a > 0$ and there is none if $x_a = 0$. Therefore, for every $0 \leq i \leq t$ satisfying $x_i > 0$, there are t such tuples. ☐

Now we can see that in $G_t(n)$, the vertices with the least degree are the ones containing the most zeroes. Geometrically, they are the corners of a simplex.

Definition 2.2. The *corners* of $G_t(n)$ are $c_0 = (n, 0, \ldots, 0), c_1 = (0, n, \ldots, 0), \ldots, c_t = (0, 0, \ldots, n)$.

Now we present a rigorous definition of the graphs $A_t(m, n)$.

Definition 2.3. The graph $A_t(m, n)$ with integer $m \geq 2$ is the subgraph of $G_t(n)$ induced by the set of every vertex $(x_0, x_1, \ldots, x_t)$ satisfying $m \nmid \binom{x_1+x_2+\ldots+x_t}{x_1,x_2,\ldots,x_t}$. A corner of $G_t(n)$ is also a corner of $A_t(m, n)$.

Proposition 2.3. *Every corner of $G_t(n)$ is in $A_t(m, n)$.*

Proof. If $(x_0, x_1, \ldots, x_t) = c_0$, then $\binom{x_1+x_2+\ldots+x_t}{x_1,x_2,\ldots,x_t} = \frac{0!}{(0!)^t} = 1$. If $(x_0, x_1, \ldots, x_t) = c_i$ where $i \geq 1$, then $\binom{x_1+x_2+\ldots+x_t}{x_1,x_2,\ldots,x_t} = \frac{n!}{n! \cdot (0!)^{t-1}} = 1$. Either way, it is not divisible by m. $\square$

Figure 2 suggests that the pattern is the most regular when m is prime and n is one less than a power of m. In this paper, we will mainly focus on this type of graphs.

Definition 2.4. For prime p and positive integer k, ${A_t}_p^k$ denotes $A_t(p, p^k - 1)$.

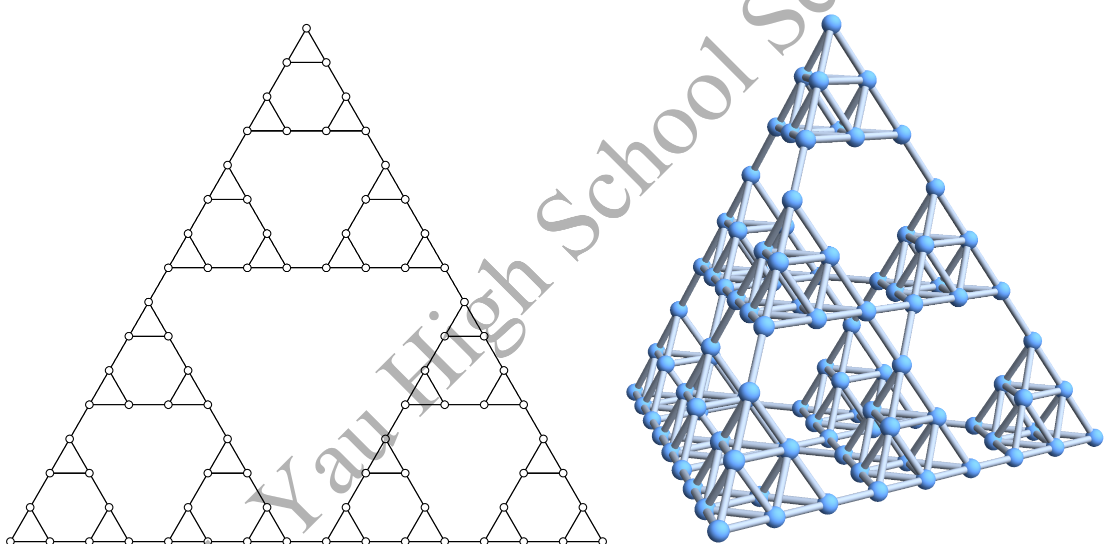


Figure 5: Graphs ${A_2}_2^4$ and ${A_3}_3^2$.

Proposition 2.4. *The graph ${A_t}_p^1$ is isomorphic to $G_t(p-1)$.*

Proof. For any vertex $(x_0, x_1, \ldots, x_t)$ in $G_t(p-1)$, $x_1 + x_2 + \ldots + x_t \leq p - 1$, so the factor p will not appear in the numerator of $\binom{x_1+x_2+\ldots+x_t}{x_1,x_2,\ldots,x_t} = \frac{(x_1+x_2+\ldots+x_t)!}{x_1!x_2!\ldots x_t!}$, so the vertex is in ${A_t}_p^1$. Therefore, ${A_t}_p^1$ is the subgraph of $G_t(p-1)$ induced by its own vertex set, so it is isomorphic to $G_t(p-1)$. $\square$

Definition 2.5. We omit t when it is equal to 2. Therefore, $G(n)$, $A(m, n)$, and A_p^k denote $G_2(n)$, $A_2(m, n)$, and ${A_2}_p^k$ respectively.

3 Symmetry and self-similarity

Since the inclusion of vertices in $A_t(m,n)$ depends on the divisibility of a multinomial coefficient by a prime number, it is useful to introduce a relevant result by Dickson [6].

Theorem 3.1. *For nonnegative integers $x_0+x_1+\ldots+x_t=s$, $\binom{s}{x_0,x_1,\ldots,x_t}$ is not divisible by prime p if and only if $s_j=\sum_{i=0}^{t}x_{ij}$ for $0\le j\le r$, where $s=\sum_{j=0}^{r}s_jp^j$, $x_i=\sum_{j=0}^{r}x_{ij}p^j$ and $0\le s_j,x_{ij}\le p-1$ for $0\le j\le r$, $0\le i\le t$.*

Applying Theorem 3.1 to $A_{t_p}^k$, we obtain the following proposition.

Proposition 3.2. *Vertex $(x_0,x_1,\ldots,x_t)$ of $G_t(p^k-1)$ is in $A_{t_p}^k$ if and only if $\sum_{i=0}^{t}x_{ij}=p-1$ for $0\le j\le k-1$, where $x_i=\sum_{j=0}^{k-1}x_{ij}p^j$, and $0\le x_{ij}\le p-1$ for $0\le j\le k-1$, $0\le i\le t$.*

Proof. Notice that $\binom{p^k-1}{x_0,x_1,x_2,\ldots,x_t}=\binom{p^k-1}{x_0}\binom{x_1+x_2+\ldots+x_t}{x_1,x_2,\ldots,x_t}$. According to Theorem 3.1, $p\nmid\binom{p^k-1}{x_0}$. Therefore, vertex $(x_0,x_1,\ldots,x_t)$ of $G_t(p^k-1)$ is in $A_{t_p}^k$ if and only if $p\nmid\binom{p^k-1}{x_0,x_1,x_2,\ldots,x_t}$. $\square$

We can see from Figure 5 that the visual representations of $A_{t_p}^k$ are highly symmetric. The following proposition shows this fact by naming all the automorphisms of $A_{t_p}^k$, which are isomorphisms from $A_{t_p}^k$ to itself. (We refrain from applying the word "symmetric" to the graphs $A_{t_p}^k$, so as not to cause confusion with the type of graphs called symmetric graphs.)

Proposition 3.3. *By changing the order of $x_0,x_1,\ldots,x_t$, we can obtain $(t+1)!$ automorphisms of $A_{t_p}^k$. No other automorphisms exists.*

Proof. Let a permutation of $\{0,1,\ldots,t\}$ be α. There are $(t+1)!$ such permutations, each of which corresponds to an isomorphism

$$f((x_0,x_1,\ldots,x_t))=\left(x_{\alpha(0)},x_{\alpha(1)},\ldots,x_{\alpha(t)}\right).$$

Now since the corners are the vertices with the least possible degree, an isomorphism must map every corner to a corner. Therefore, the number of isomorphisms is no more than the number of permutations of corners, which is $(t+1)!$. This completes the proof. $\square$

The following corollary is useful when we prove the Hamiltonicity of $A_{t_p}^k$ in Section 7.

Corollary. *For any two pairs of corners (c_i,c_j) and $(c_{i'},c_{j'})$ such that $i\ne j$ and $i'\ne j'$, there is an isomorphism that maps c_i to $c_{i'}$ and c_j to $c_{j'}$.*

Self-similarity is what defines a fractal. The Sierpiński triangle is a fractal because it can be divided into three smaller triangles, each of which is similar to the original one. The drawings of A_2^k approximate the Sierpiński triangle, and the (t-dimensional) geometrical representations of the graphs $A_{t_p}^k$ also show similar structures. Therefore, it is reasonable to suggest that, although a finite, discrete object cannot be partitioned into parts that are identical to itself, $A_{t_p}^k$ can still possess some form of self-similarity. In the following definition, we propose a partition of the vertex set of $A_{t_p}^k$ such that every part of such a partition induces a subgraph that is isomorphic to $A_{t_p}^{k-1}$. Such a partition can prove useful when investigating other problems, as we will show in Sections 4, 6 and 7.

Definition 3.1. For nonnegative integers $x_0'+x_1'+\ldots+x_t'=p-1$, a block $B_{x_0',x_1',\ldots,x_t'}$ of $P=A_{t_p}^k$ is the subgraph of P induced by the set of every vertex $(x_0,x_1,\ldots,x_t)$ such that $\left\lfloor\frac{x_i}{p^{k-1}}\right\rfloor=x_i'$ for $0\le i\le t$.

Proposition 3.4. *Every vertex in $A_{t_p}^k$ is in exactly one block, and every block of $A_{t_p}^k$ is isomorphic to $A_{t_p}^{k-1}$.*

Proof. According to Proposition 3.2, vertex $(x_0, x_1, \ldots, x_t)$ of $G_t(p^k - 1)$ is in $A_{t_p}^k$ if and only if $\sum_{i=0}^{t} \left\lfloor \frac{x_i}{p^{k-1}} \right\rfloor = p - 1$ and $(x_0 \bmod p^{k-1}, x_1 \bmod p^{k-1}, \ldots, x_t \bmod p^{k-1})$ is in $A_{t_p}^{k-1}$.

By uniqueness of the division algorithm, no vertex in $A_{t_p}^k$ can be in two different blocks. And the first condition implies that every vertex $(x_0, x_1, \ldots, x_t)$ in $A_{t_p}^k$ is in block $B_{\left\lfloor \frac{x_0}{p^{k-1}} \right\rfloor, \left\lfloor \frac{x_1}{p^{k-1}} \right\rfloor, \ldots, \left\lfloor \frac{x_t}{p^{k-1}} \right\rfloor}$.

By the second condition, the vertex set of $B_{x'_0, x'_1, \ldots, x'_t}$ is $\{(x'_0 p^{k-1} + r_0, x'_1 p^{k-1} + r_1, \ldots, x'_t p^{k-1} + r_t) | (r_0, r_1, \ldots, r_t) \in A_{t_p}^1\}$. Also notice that $(x'_0 p^{k-1} + r_0, x'_1 p^{k-1} + r_1, \ldots, x'_t p^{k-1} + r_t)$ and $(x'_0 p^{k-1} + s_0, x'_1 p^{k-1} + s_1, \ldots, x'_t p^{k-1} + s_t)$ are adjacent if and only if $(r_0, r_1, \ldots, r_t)$ and $(s_0, s_1, \ldots, s_t)$ are adjacent. Therefore, $B_{x'_0, x'_1, \ldots, x'_t}$ is isomorphic to $A_{t_p}^1$. □

Definition 3.2. For $G = A_{t_p}^k$, two different blocks B_1 and B_2 are b-adjacent when there exists edge $uv \in G$ such that $u \in B_1$ and $v \in B_2$. Such an edge is called a *bridge* between B_1 and B_2.

Proposition 3.5. *Two different blocks $B_{x'_0, x'_1, \ldots, x'_t}$ and $B_{y'_0, y'_1, \ldots, y'_t}$ are b-adjacent if and only if $\frac{1}{2} \sum_{i=0}^{t} |x'_i - y'_i| = 1$. There is exactly one bridge between two such blocks, and its endpoints are corners of each block.*

Proof. First, suppose that there exist adjacent vertices $(x_0, x_1, \ldots, x_t) \in B_{x'_0, x'_1, \ldots, x'_t}$ and $(y_0, y_1, \ldots, y_t) \in B_{y'_0, y'_1, \ldots, y'_t}$. Therefore, there exists integers $a, b \in \{0, 1, \ldots, t\}$, $a \neq b$ such that $x_a = y_a + 1$, $y_b = x_b + 1$, and $x_c = y_c$ for $c \in \{0, 1, \ldots, t\} - \{a, b\}$. Therefore, $\frac{1}{2} \sum_{i=0}^{t} |x'_i - y'_i| = \frac{1}{2} \sum_{i=0}^{t} \left| \left\lfloor \frac{x_i}{p^{k-1}} \right\rfloor - \left\lfloor \frac{y_i}{p^{k-1}} \right\rfloor \right| \leq \frac{1}{2} \sum_{i=0}^{t} |x_i - y_i| = 1$. Since $B_{x'_0, x'_1, \ldots, x'_t}$ and $B_{y'_0, y'_1, \ldots, y'_t}$ are different blocks, they can only be b-adjacent.

Second, suppose that $B_{x'_0, x'_1, \ldots, x'_t}$ and $B_{y'_0, y'_1, \ldots, y'_t}$ are b-adjacent blocks in $A_{t_p}^k$. Therefore there exists integers $a, b \in \{0, 1, \ldots, t\}$, $a \neq b$ such that $x'_a = y'_a + 1$, $y'_b = x'_b + 1$, and $x'_c = y'_c$ for $c \in \{0, 1, \ldots, t\} - \{a, b\}$. Therefore, for vertices $(x_0, x_1, \ldots, x_t) \in B_{x'_0, x'_1, \ldots, x'_t}$ and $(y_0, y_1, \ldots, y_t) \in B_{y'_0, y'_1, \ldots, y'_t}$, $x_a > y_a$, $y_b > x_b$. Suppose they are adjacent. Then $x_a = y_a + 1$, $y_b = x_b + 1$. Now $(y'_a + 1) p^{k-1} + (x'_a \bmod p^{k-1}) = y'_a p^{k-1} + (y'_a \bmod p^{k-1}) + 1$, so $x'_a \bmod p^{k-1} = 0$, $y'_a \bmod p^{k-1} = p^{k-1} - 1$. Similarly $y'_b \bmod p^{k-1} = 0$, $x'_b \bmod p^{k-1} = p^{k-1} - 1$. By Proposition 3.2, $\sum_{i=0}^{t} (x'_i \bmod p^{k-1}) = \sum_{i=0}^{t} (y'_i \bmod p^{k-1}) = p^{k-1} - 1$. So $x_c = x'_c p^{k-1}$ for $0 \leq c \leq t$, $c \neq b$, $x_b = x'_b p^{k-1} + p^{k-1} - 1$, and $y_c = y'_c p^{k-1}$ for $0 \leq c \leq t$, $c \neq a$, $y_a = y'_a p^{k-1} + p^{k-1} - 1$. And it is easy to see that $(x_0, x_1, \ldots, x_t)$ and $(y_0, y_1, \ldots, y_t)$ are indeed adjacent. Thus there is exactly one bridge between $B_{x'_0, x'_1, \ldots, x'_t}$ and $B_{y'_0, y'_1, \ldots, y'_t}$, and its endpoints are corners of each block. □

Definition 3.3. Let the block-graph of $G = A_{t_p}^k$ be the graph G' whose vertices are the blocks of G, and two vertices are adjacent if they are b-adjacent.

Proposition 3.6. *For $G = A_{t_p}^k$, G' is isomorphic to $G_t(p-1)$, and thus to $A_{t_p}^1$.*

Proof. By Propositions 2.4 and 3.5, $f(B_{x'_0, x'_1, \ldots, x'_t}) = (x'_0, x'_1, \ldots, x'_t)$ is an isomorphism from G' to $A_{t_p}^1$. □

4 Number of vertices and edges

Proposition 4.1. *The graph $G_t(n)$ has $\binom{n+t}{t}$ vertices and $\frac{t(t+1)}{2}\binom{n+t-1}{t}$ edges.*

Proof. The number of vertices of $G_t(n)$ is equal to the number of ways to distribute n identical objects into $t+1$ distinguishable containers, which is well understood to be $\binom{n+t}{t}$.

By Proposition 2.2, $\deg((x_0, x_1, \ldots, x_t)) = t\sum_{i=0}^{t} \operatorname{sgn}(x_i)$. Therefore,

$$\begin{aligned} |E(G_t(n))| &= \frac{1}{2} \sum_{(x_0,x_1,\ldots,x_t)\in G_t(n)} \left(t \sum_{i=0}^{t} \operatorname{sgn}(x_i) \right) \\ &= \frac{t(t+1)}{2} \sum_{(x_0,x_1,\ldots,x_t)\in G_t(n)} \operatorname{sgn}(x_0) \\ &= \frac{t(t+1)}{2} \binom{n+t-1}{t} \end{aligned}$$

The last step of the equation above derives from the fact that the number of vertices $(x_0, x_1, \ldots, x_t)$ in $G_t(n)$ where x_0 is nonzero is equal to the number of ways to distribute $n-1$ identical objects (the missing one is reserved for x_0) into $t+1$ distinguishable containers. $\square$

Proposition 4.2. *The graph $A_{t_p}^{k}$ has $\binom{p+t-1}{t}^k$ vertices and $\frac{t(t+1)\binom{p+t-2}{t}\left(\binom{p+t-1}{t}^k-1\right)}{2\left(\binom{p+t-1}{t}-1\right)}$ edges.*

Proof. By Proposition 4.1, the proposition is true for $k=1$. Assume that it is true for some k. Then by Propositions 3.4, 3.6 and 4.1, $|V(A_{t_p}^{k+1})| = \binom{p+t-1}{t}|V(A_{t_p}^{k})| = \binom{p+t-1}{t}^{k+1}$, and $|E(A_{t_p}^{k+1})| = \binom{p+t-1}{t}|E(A_{t_p}^{k})| + \frac{t(t+1)}{2}\binom{p+t-2}{t} = \frac{t(t+1)\binom{p+t-2}{t}\left(\binom{p+t-1}{t}^{k+1}-1\right)}{2\left(\binom{p+t-1}{t}-1\right)}$. By induction, this proposition holds for every positive integer k. $\square$

5 Graphs A_{t2}^{k} and the Sierpiński graphs

The Sierpiński graphs, introduced by Klavžar and Milutinović in [14], were motivated by a topological construction called the Lipscomb space [15]. It has been proven in [14] that the Sierpiński graph S_3^n is isomorphic to the Tower of Hanoi graph TH_n, and is therefore isomorphic to A_2^n. In this section, we generalize this result to prove the following theorem.

Theorem 5.1. *The graph A_{t2}^{k} is isomorphic to the Sierpiński graph S_{t+1}^{k}.*

Proof. By Proposition 3.2, vertex $(x_0, x_1, \ldots, x_t)$ of $G_t(2^k-1)$ is in A_{t2}^{k} if and only if $\sum_{i=0}^{t} x_{ij} = 1$ for $0 \le j \le k-1$, where $x_i = \sum_{j=0}^{k-1} x_{ij}2^j$ and $0 \le x_{ij} \le 1 \ \forall\ 0 \le j \le k-1$, $0 \le i \le t$. Therefore, let $f((x_0, x_1, \ldots, x_t)) = (s_1, s_2, \ldots, s_k)$, such that $x_{s_d d-1} = 1 \ \forall\ 1 \le d \le k$, and f is a bijection between $V(A_{t2}^{k})$ and $V(S_{t+1}^{k})$. Now we prove that f is adjacency-preserving.

By Proposition 2.1, two vertices $x = (x_0, x_1, \ldots, x_t)$ and $y = (y_0, y_1, \ldots, y_t)$ of A_{t2}^{k} are adjacent if and only if there exist integers $a, b \in \{0, 1, \ldots, t\}$, $a \ne b$ such that $x_a = y_a + 1$, $y_b = x_b + 1$, and $x_c = y_c$ for $c \in \{0, 1, \ldots, t\} - \{a, b\}$. Now $x_a > 0$, so let δ be the least integer such that $2^\delta \nmid x_a$, and let $s = (s_1, s_2, \ldots, s_k) = f(x)$ and $t = (t_1, t_2, \ldots, t_k) = f(y)$. Then $s_\delta = a$, $t_\delta \ne a$, and $s_d \ne a$, $t_d = a$ $\forall\ 1 \le d \le \delta - 1$. Now for any $c \in \{0, 1, \ldots, t\} - \{a, b\}$, $x_c = y_c$, so for any $1 \le d \le k$, $s_d = c$ if and only if $t_d = c$. Therefore, $t_\delta = b$ and $s_d = b$ for $1 \le d \le \delta - 1$, so s and t are adjacent in S_{t+1}^{k}. Finally, according to [13], $|E(S_{t+1}^{0})| = 0$ and $|E(S_{t+1}^{k+1})| = (t+1)|E(S_{t+1}^{k})| + \binom{t+1}{2}$, and by Proposition 4.2, $|E(A_{t2}^{k})| = \frac{(t+1)((t+1)^k-1)}{2}$, so S_{t+1}^{k} and $|E(A_{t2}^{k})|$ have the same number of edges, and our proof is complete. $\square$

6 Diameter

Chen et al. [3] proved that $\operatorname{diam}(S_k^n) = 2^n - 1$. In this section, we will extend this result by proving that $\operatorname{diam}(A(m,n)) = n$ if $n = rp^k - 1$ $(1 \le r \le p-1)$ where prime p divides m, and that $\operatorname{diam}(A_{tp}^k) = p^k - 1$. The former contains the result in [3] while the latter contains the $k = 3$ case.

Lemma 6.1. *In the graph $G_t(n)$, $d(x,y) = \frac{1}{2}\sum_{i=0}^{t} |x_i - y_i|$, where $x = (x_0, x_1, \ldots, x_t)$ and $y = (y_0, y_1, \ldots, y_t)$.*

Proof. By definition, if $\frac{1}{2}\sum_{i=0}^{t} |x_i - y_i| = 1$, then $d(x,y) = 1$. Assume that if $\frac{1}{2}\sum_{i=0}^{t} |x_i - y_i| = k$, then $d(x,y) = k$. Then if $\frac{1}{2}\sum_{i=0}^{t} |x_i - y_i| = k+1$, then there exists $a, b \in \{0, 1, \ldots, t\}$, $a \ne b$, such that $x_a > y_a$, $x_b < y_b$. Consider $z = (z_0, z_1, \ldots, z_t)$ where $z_a = x_a - 1$, $z_b = x_b + 1$, $z_c = x_c$ for $c \in \{0, 1, \ldots, t\} - \{a, b\}$. First observe that $\frac{1}{2}\sum_{i=0}^{t} |z_i - y_i| = \frac{1}{2}\sum_{i=0}^{t} |x_i - y_i| + \frac{1}{2}(|z_a - y_a| - |x_a - y_a| + |z_b - y_b| - |x_b - y_b|) = \frac{1}{2}\sum_{i=0}^{t} |x_i - y_i| + \frac{1}{2}((z_a - x_a) - (z_b - x_b)) = (k+1) - 1 = k$. By the induction hypothesis $d(z,y) = k$. Therefore, $d(x,y) \le d(x,z) + d(z,y) = k+1$.

Then let $d = d(x,y)$ and let a shortest path between x and y be $x^0, x^1, \ldots, x^d$ where $x^0 = x$, $x^d = y$, and $x^j = (x_0^j, x_1^j, \ldots, x_t^j)$ for $0 \le j \le d$. Then $d(x,y) = d = \sum_{j=0}^{d-1}\left(\frac{1}{2}\sum_{i=0}^{t-1} |x_i^j - x_i^{j+1}|\right) = \frac{1}{2}\sum_{i=0}^{t-1}\sum_{j=0}^{d-1} |x_i^j - x_i^{j+1}| \ge \frac{1}{2}\sum_{i=0}^{t-1} |x_i^0 - x_i^d| = \frac{1}{2}\sum_{i=0}^{t-1} |x_i - y_i| = k+1$.

Therefore, $d(x,y) = k+1$, and by induction the proof is complete. □

Lemma 6.2. *The diameter of $G_t(n)$ is n.*

Proof. By Lemma 6.1, for any two vertices $x, y \in G_t(n)$,

$$d(x,y) = \frac{1}{2}\sum_{i=0}^{t} |x_i - y_i| \le \frac{1}{2}\sum_{i=0}^{t} (x_i + y_i) = n.$$

Also, the distance between two different corners is at least n. Therefore, $\operatorname{diam}(G_t(n)) = n$. □

Theorem 6.3. *The diameter of A_{tp}^k is $p^k - 1$.*

Proof. When $k = 1$, the theorem is true by Lemma 6.2. Assume that it is true for some k. Now we consider the case of $G = A_{tp}^{k+1}$.

Let $x \in B_{x'}$ and $y \in B_{y'}$ be two vertices. Let $P = B_0, B_1, \ldots, B_l$ where $B_0 = B_{x'}$ and $B_l = B_{y'}$ be a shortest path in G' between $B_{x'}$ and $B_{y'}$. By Lemma 6.2 and Proposition 3.6, $l \le p - 1$. Let $v_0v_1, v_2v_3, \ldots, v_{2l-2}v_{2l-1}$ be the bridges between consecutive blocks in P, such that $v_{2i} \in B_i$ and $v_{2i+1} \in B_{i+1}$ for $0 \le i \le l-1$. Now observe that

$$\begin{aligned}
d(x,y) &\le d(x,v_0) + d(v_0,v_1) + d(v_1,v_2) + d(v_2,v_3) + \ldots + d(v_{2l-2},v_{2l-1}) + d(v_{2l-1},y) \\
&= d(x,v_0) + d(v_1,v_2) + \ldots + d(v_{2l-1},y) + l \\
&\le (l+1)\operatorname{diam}(A_{tp}^k) + l \\
&\le p(p^k - 1) + p - 1 \\
&= p^{k+1} - 1
\end{aligned}$$

Finally, by Lemma 6.2, $\operatorname{diam}(A_{tp}^{k+1}) \ge p^{k+1} - 1$. This completes the proof. □

Now when $t = 2$, $A_{tp}^k = A_p^k$ is planar, and we can exploit its planarity to prove a more general result. We start with a lemma.

Lemma 6.4. *Of vertices (x_0, x_1, x_2-1), (x_0, x_1-1, x_2), (x_0-1, x_1, x_2) in $G(n)$, if one of them is in $A(m,n)$, then at least one of the other two is in $A(m,n)$.*

Proof. Consider (x_0, x_1-1, x_2) and (x_0, x_1, x_2-1). Since $\binom{(x_1-1)+x_2}{x_1-1}+\binom{x_1+(x_2-1)}{x_1}=\binom{x_1+x_2}{x_1}$, if neither of them is in $A(m,n)$, then (x_0-1, x_1, x_2) is not in $A(m,n)$. If exactly one of them is in $A(m,n)$, then (x_0-1, x_1, x_2) is in $A(m,n)$. If both of them are in $A(m,n)$, then the lemma holds regardless of whether (x_0-1, x_1, x_2) is in $A(m,n)$ or not. □

Theorem 6.5. *The diameter of $G = A(m,n)$ is equal to n if $(0,n,0),(0,n-1,1),\ldots,\ (0,0,n) \in G$.*

Proof. By Lemma 6.2, $\operatorname{diam}(A(m,n)) \geq n$, so we only need to prove that $d((x_0,x_1,x_2),\ (y_0,y_1,y_2)) \leq n$ for every $(x_0,x_1,x_2),(y_0,y_1,y_2) \in G$. Notice that when $x_1 = 0$ or $x_2 = 0$, $(x_0,x_1,x_2) \in G$, and $(0,n,0),(0,n-1,1),\ldots,\ (0,0,n) \in G$, so every vertex containing 0 is in G.

Let P_{0x} be a maximal path $(x_0^0,x_1^0,x_2^0),(x_0^1,x_1^1,x_2^1),\ldots,(x_0^l,x_1^l,x_2^l)$ constructed with the following rules:

1) $(x_0^0,x_1^0,x_2^0) = (x_0,x_1,x_2)$;

2) $x_0^{i+1} = x_0^i + 1$ for $0 \leq i \leq l-1$.

By Lemma 6.4, 2) can be achieved when both $(x_0^i+1, x_1^i-1, x_2^i)$ and $(x_0^i+1, x_1^i, x_2^i-1)$ are in $G(n)$. If exactly one of them is in $G(n)$, it means that exactly one of x_1^i or x_2^i is 0. Therefore the one candidate that is in $G(n)$ also contains 0, and is therefore in G. Therefore, the process stops only when neither candidate is in $G(n)$, so $(x_0^l,x_1^l,x_2^l) = c_0$.

Similarly, we construct paths P_{1x} and P_{2x}, which end in c_1 and c_2 respectively. And similarly, we start from (y_0,y_1,y_2) to construct paths P_{0y}, P_{1y} and P_{2y}.

Since every vertex (x_0^i,x_1^i,x_2^i) in P_{0x} satisfies $x_0^i = x_0 + i$, and every vertex (x_0^j,x_1^j,x_2^j) in P_{1x} satisfies $x_0^j \leq x_0$, P_{0x} and P_{1x} do not intersect except at the common starting vertex. Similarly, no two of P_{0x}, P_{1x} and P_{2x} intersect except at the common starting vertex. Therefore, these three paths partition G into three non-overlapping regions, such that every vertex in G is either in one of these regions or on one of these paths.

The length of any of these paths is less than or equal to n, so if (y_0,y_1,y_2) is on one of these paths then the theorem holds. If not, then WLOG let (y_0,y_1,y_2) be in the region confined by P_{1x} and P_{2x}. Then P_{0y} intersects with one of P_{1x} and P_{2x} (or with both, if it passes through (x_0,x_1,x_2)). WLOG let P_{0y} intersect with P_{1x}. Let the first vertex in P_{0y} that is also in P_{1x} be $(y_0^j,y_1^j,y_2^j) = (x_0^i,x_1^i,x_2^i)$. Since $(x_0^i,x_1^i,x_2^i) = (y_0^j,y_1^j,y_2^j) \in G$, $d((x_0,x_1,x_2),(y_0,y_1,y_2)) \leq d((x_0,x_1,x_2),(x_0^i,x_1^i,x_2^i)) + d((y_0^j,y_1^j,y_2^j),(y_0,y_1,y_2)) = i + j \leq x_1 + i + y_0 + j = x_1^i + y_0^j = x_1^i + x_0^i \leq x_0^i + x_1^i + x_2^i = n$. □

The process elaborated in the proof of Theorem 6.5 is illustrated by Figure 6. The two arbitrarily chosen vertices and the paths they generate are highlighted with red (dashed) and blue (dotted) respectively. Overlapping edges are drawn in purple (dash-dotted). The desired path is marked by the thick edges.

Given m, the following proposition can be used to find infinitely many values of n for which $\operatorname{diam}(A(m,n)) = n$.

Proposition 6.6. *For prime $p|m$ and $n = rp^k - 1$ $(1 \leq r \leq p-1)$,* $\operatorname{diam}(A(m,n)) = n$.

Proof. By Theorem 3.1, for $0 \leq x \leq rp^k - 1$, we have $p \nmid \binom{rp^k-1}{x}$, so $m \nmid \binom{rp^k-1}{x}$. Therefore, we have $(0,n,0),(0,n-1,1),\ldots,(0,0,n) \in G$, and by Theorem 6.5, $\operatorname{diam}(A(m,n)) = n$. □

However, the analogue of Theorem 6.5 for $t \geq 3$ does not hold. Indeed, we make the following conjecture.

Conjecture. *For any $t \geq 3$ and composite m, there exists positive integer N such that the diameter of $A_t(m,n)$ is greater than n $\forall$ $n > N$.*

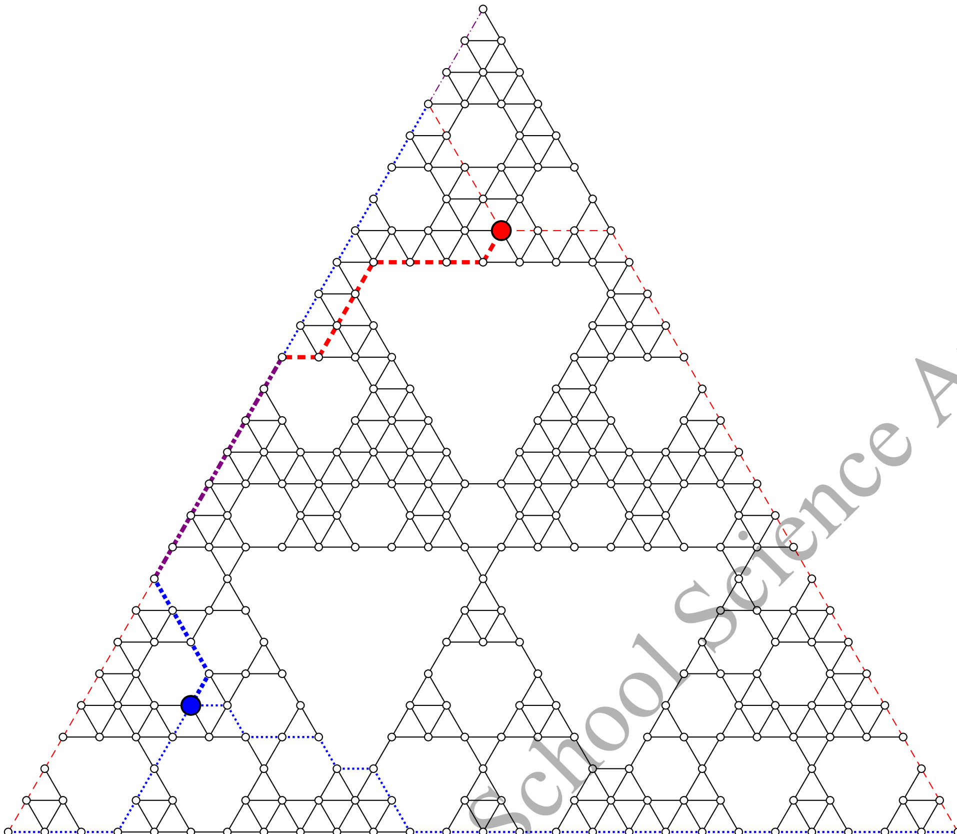


Figure 6: Illustration for proof of Theorem 6.5. The graph used here is $A(6, 26)$.

7 Hamiltonicity

In [20], Teguia and Godbole proved that the Sierpiński gasket graphs are Hamiltonian, and proposed as an open problem the Hamiltonicity of the generalized Sierpiński gasket graphs generated by Pascal's triangle modulo a prime greater than 2. In this section, we will prove the existence of Hamiltonian paths and cycles that satisfy a certain property in $A_{t_p}^{k}$, and in the next section, we will apply our result to prove that these generalized Sierpiński gasket graphs are indeed Hamiltonian. By the isomorphism between $A_{t_2}^{k}$ and S_{t+1}^{k}, our result in this section also partly contains the result by Klavžar and Milutinović [14] that the Sierpiński graphs S_k^n are Hamiltonian.

Definition 7.1. For tuple of nonnegative integers $x' = (x'_0, x'_1, \ldots, x'_t)$, $F(x')$ is defined to be the set $\{(x'_0 - 1, x'_1, \ldots, x'_t), (x'_0, x'_1 - 1, \ldots, x'_t), \ldots, (x'_0, x'_1, \ldots, x'_t - 1)\}$.

A set S is called a funnel if there exists some $x' = (x'_0, x'_1, \ldots, x'_t)$ where $t \geq 2$ such that $S = F(x')$.

For some funnel $S = F((x'_0, x'_1, \ldots, x'_t))$, the element $(x_0, x_1, \ldots, x_t) \in S$ where $x_a = x'_a - 1$ and $x_i = x'_i \; \forall \; i \in \{0, 1, \ldots, t\} - \{a\}$ is denoted as S_a.

A set of blocks in some $G = A_{t_p}^{k}$ is a block-funnel if it is mapped to a funnel by an isomorphism from G' to $A_{t_p}^{1}$.

We will consider the intersections between funnels and the vertex sets of the graphs $G_t(n)$ and $A_{t_p}^{k}$. Therefore, it is useful to note the following lemmas.

Lemma 7.1. *If x and y are two adjacent vertices of $G_t(n)$, then there is exactly one funnel containing both x and y.*

Proof. Let $x = (x_0, x_1, \ldots, x_t)$ and $y = (y_0, y_1, \ldots, y_t)$. Assume that $x_a = y_a + 1$, $y_b = x_b + 1$, and $x_c = y_c$ for $c \in \{0, 1, \ldots, t\} - \{a, b\}$. Therefore, the funnel $S = F((x'_0, x'_1, \ldots, x'_t))$ where $x'_a = x_a = y_a + 1$, $x'_b = y_b = x_b + 1$, and $x'_c = x_c = y_c$ for $c \in \{0, 1, \ldots, t\} - \{a, b\}$ contains both x and y, since $x = S_b$ and $y = S_a$.

Now suppose for contradiction that funnel $T = F((y'_0, y'_1, \ldots, y'_t))$ also includes both x and y, and that $T \neq S$. Assume that $x = T_{b'}$ and $y = T_{a'}$. Since $T \neq S$, we have $a' \neq a$. Therefore, $y_a + 1 = x_a \leq y'_a = y_a$, which is a contradiction. This completes the proof. □

Lemma 7.2. *For the corner c_i of $G_t(n)$, any funnel that contains it must contain at most one other vertex of $G_t(n)$.*

Proof. Assume that funnel $S = F(x')$ contains c_i. First observe that the number of elements of S that are also vertices in $G_t(n)$ is equal to the number of positive integers in x', which is at most one more than that of c_i, which has only one positive integer. Therefore, $|S \cap G_t(n)| \leq 2$. □

Lemma 7.3. *If $x = (x_0, x_1, \ldots, x_t)$, $y = (y_0, y_1, \ldots, y_t)$ and $z = (z_0, z_1, \ldots, z_t)$ are different elements of $S = F((x'_0, x'_1, \ldots, x'_t))$ and c is an integer such that $x_c = y_c$, then $z_c \leq x_c = y_c$.*

Proof. Suppose for contradiction that $z_c > x_c$. Since $x'_c \geq z_c$, we have $x'_c > x_c$ and $x'_c > y_c$, which means that $x = y = S_c$, which is a contradiction. □

The funnel is thus named because when $t = 2$, it (or rather the subgraph of $G_t(n)$ it induces) looks like a funnel. But note that by the automorphic properties of $G_t(n)$ and A_{tp}^{k}, the "bottom" of a funnel (that is, S_0) does not distinguish itself from the other elements of the funnel.

Definition 7.2. A path or cycle in $G_t(n)$ or A_{tp}^{k} (or in the block-graph G') is funnel-free if it does not contain two different unordered pairs of consecutive vertices such that their union is the subset of a funnel (or block-funnel). (The two pairs need not be disjoint - for example, (u, v) and (v, w) also count.)

Note that a funnel is mapped to a new funnel when the numbers in every one of its elements are permuted in a certain way. A subset of a funnel is mapped to the subset of a new funnel when a certain index with the same value in every one of its elements is deleted.

Example 7.1. Let $S = F((x_0, x_1, x_2, x_3))$, then $\{(x_2, x_0, x_1, x_3 - 1), (x_2 - 1, x_0, x_1, x_3), (x_2, x_0, x_1 - 1, x_3), (x_2, x_0 - 1, x_1, x_3)\}$ is a funnel. Also, $\{(x_0, x_1, x_2, x_3 - 1), (x_0, x_1, x_2 - 1, x_3), (x_0 - 1, x_1, x_2, x_3)\} \subset S$ can be mapped to $\{(x_0, x_2, x_3 - 1), (x_0, x_2 - 1, x_3), (x_0 - 1, x_2, x_3)\}$, which is also the subset of a funnel (though not a proper subset).

Lemma 7.4. *In the graph $G_t(n)$, there exist funnel-free Hamiltonian paths between c_i and c_j for $0 \leq i < j \leq t$.*

Proof. When $t = 1$, the graph is itself a path, so its only Hamiltonian path is funnel-free. When $n = 0$, the graph has a Hamiltonian path with a length of 0, which is funnel-free. (These degenerate cases are discussed here only to simplify the base case proof.)

Therefore, the lemma holds when $t + n = 1$. Suppose that it holds when $t + n = s$. Then we consider the case when $t + n = s + 1$. If $t = 1$ or $n = 0$, then the lemma holds. Therefore, assume that $t > 1$ and $n > 0$.

Let G_1 and G_2 be subgraphs of $G_t(n)$ induced respectively by the set of all vertices $(x_0, x_1, \ldots, x_t)$ where $x_0 > 0$ and the set of all vertices $(y_0, y_1, \ldots, y_t)$ where $y_0 = 0$. Isomorphisms between G_1 and

$G_t(n-1)$ and between G_2 and $G_{t-1}(n)$ are given by $f((x_0, x_1, \ldots, x_t)) = (x_0 - 1, x_1, \ldots, x_t)$ and $g((y_0, y_1, \ldots, y_t)) = (y_1, \ldots, y_t)$. Notice that the intersection of a funnel with the vertex set of one of the two lesser simplex grid graphs is mapped to the intersection of a funnel with the vertex set of the corresponding subgraph by these isomorphisms.

Since the lemma holds when $t + n = s$, both $G_t(n-1)$ and $G_{t-1}(n)$ have funnel-free Hamiltonian paths between any two corners. Let P_1 be a funnel-free Hamiltonian path of G_1 between $(n, 0, \ldots, 0)$ and $(1, n-1, \ldots, 0)$ and let P_2 be a Hamiltonian path of G_2 between $(0, n, \ldots, 0)$ and $(0, 0, \ldots, n)$. We concatenate these two paths to get P, a Hamiltonian path of $G_t(n)$.

Suppose for contradiction that P is not funnel-free. Then let (u_1, u_2) and (v_1, v_2) be the two pairs of consecutive vertices in P whose union is the subset of a funnel. By Lemma 7.2, no such pair contains $(0, n, \ldots, 0)$, so each is either a subset of $V(G_1)$ or a subset of $V(G_2)$. By our hypothesis, the two pairs cannot both be subsets of $V(G_1)$ or of $V(G_2)$. Therefore, one such pair is a subset of $V(G_1)$ and the other one a subset of $V(G_2)$. However, according to Lemma 7.3, if two vertices in a funnel are in $V(G_2)$, then no other vertex in the funnel is in $V(G_1)$, which is a contradiction. Therefore, P is funnel-free, and this completes the proof. $\square$

The following proof is very similar to that of Lemma 7.4, and certain details are not repeated here.

Lemma 7.5. *In the graph $G_t(n)$, there exists a funnel-free Hamiltonian cycle.*

Proof. As in the proof of Lemma 7.4, let G_1 and G_2 be subgraphs of $G_t(n)$ induced respectively by the set of all vertices $(x_0, x_1, \ldots, x_t)$ where $x_0 > 0$ and the set of all vertices $(y_0, y_1, \ldots, y_t)$ where $y_0 = 0$. Let P_1 be a funnel-free Hamiltonian path of G_1 between $(1, n-1, \ldots, 0)$ and $(1, 0, \ldots, n-1)$ and let P_2 be a Hamiltonian path of G_2 between $(0, n, \ldots, 0)$ and $(0, 0, \ldots, n)$. We concatenate these two paths to get a Hamiltonian cycle C of $G_t(n)$.

Suppose that C is not funnel-free. A contradiction can be obtained with an argument similar to that used in the proof of Lemma 7.4, completing this proof. $\square$

Lemma 7.6. *For $G = {A_{t}}_p^k$ and funnel S, the elements of $S' = S \cap G$ where $|S'| \geq 3$ are either all in the same block or are in $|S'|$ different blocks. In the latter case, the set of the corresponding blocks is the subset of a block-funnel in the block-graph G'.*

Proof. Let $S = F((x'_0, x'_1, \ldots, x'_t))$. Assume that $S_a = (y_0, y_1, \ldots, y_t)$ and $S_b = (x_0, x_1, \ldots, x_t)$ are in S' and are in different blocks. Since S_a and S_b are adjacent, the blocks are b-adjacent, which means that $\left\lfloor \frac{y_a}{p^{k-1}} \right\rfloor + 1 = \left\lfloor \frac{x_a}{p^{k-1}} \right\rfloor$ and $\left\lfloor \frac{y_b}{p^{k-1}} \right\rfloor = \left\lfloor \frac{x_b}{p^{k-1}} \right\rfloor + 1$. Therefore, $\sum_{i=0}^{t} \left\lfloor \frac{x'_i}{p^{k-1}} \right\rfloor = \sum_{i=0}^{t} \left\lfloor \frac{x_i}{p^{k-1}} \right\rfloor + 1 = p = \sum_{i=0}^{t} \frac{x'_i}{p^{k-1}}$, so $p^{k-1} \mid x'_i \ \forall\, 0 \leq i \leq t$. So if $S_{a'}$ and $S_{b'}$ are two elements of S', then $\left\lfloor \frac{x'_{a'}-1}{p^{k-1}} \right\rfloor < \left\lfloor \frac{x'_{a'}}{p^{k-1}} \right\rfloor$, which means that $S_{a'}$ and $S_{b'}$ are in different blocks.

Now observe that by an isomorphism from G' to ${A_{t}}_p^1$, the block that contains $S_{a'}$ is mapped to an element of $F\left(\left(\left\lfloor \frac{x'_0}{p^{k-1}} \right\rfloor, \left\lfloor \frac{x'_1}{p^{k-1}} \right\rfloor, \ldots, \left\lfloor \frac{x'_t}{p^{k-1}} \right\rfloor\right)\right)$. $\square$

Lemma 7.7. *In ${A_{t}}_p^k$ there exist funnel-free Hamiltonian paths between c_i and c_j for $0 \leq i < j \leq t$.*

Proof. We proceed with induction. By Lemma 7.4, this lemma holds when $k = 1$. Assume that it holds for ${A_{t}}_p^k$. Let $G = {A_{t}}_p^{k+1}$, and for every block there exists funnel-free Hamiltonian paths between c_i and c_j for $0 \leq i < j \leq t$. Let P' be a funnel-free Hamiltonian path of the block-graph G' between $B_{p-1,0,\ldots,0}$ and $B_{0,0,\ldots,p-1}$. Let P be a Hamiltonian path of G between $(p^k - 1, 0, \ldots, 0)$ and $(0, 0, \ldots, p^k - 1)$, obtained by concatenating funnel-free Hamiltonian paths of the blocks in the order of P'.

Suppose for contradiction that P is not funnel-free. Then let (u_1, u_2) and (v_1, v_2) be the two pairs of consecutive vertices in P whose union is the subset of a funnel. The vertices cannot all be in the same block, and by Lemma 7.6, P' is not funnel-free, which is a contradiction. $\square$

Theorem 7.8. *There is a funnel-free Hamiltonian cycle in* $A_{t_p}^{k}$.

Proof. Let $G = A_{t_p}^{k}$, and let C' be a funnel-free Hamiltonian cycle of the block-graph G'. Let C be a Hamiltonian path of G obtained by concatenating funnel-free Hamiltonian paths of the blocks in the order of C'.

Suppose that C is not funnel-free. A contradiction can be obtained with an argument similar to that used in the proof of Lemma 7.7, completing this proof. $\square$

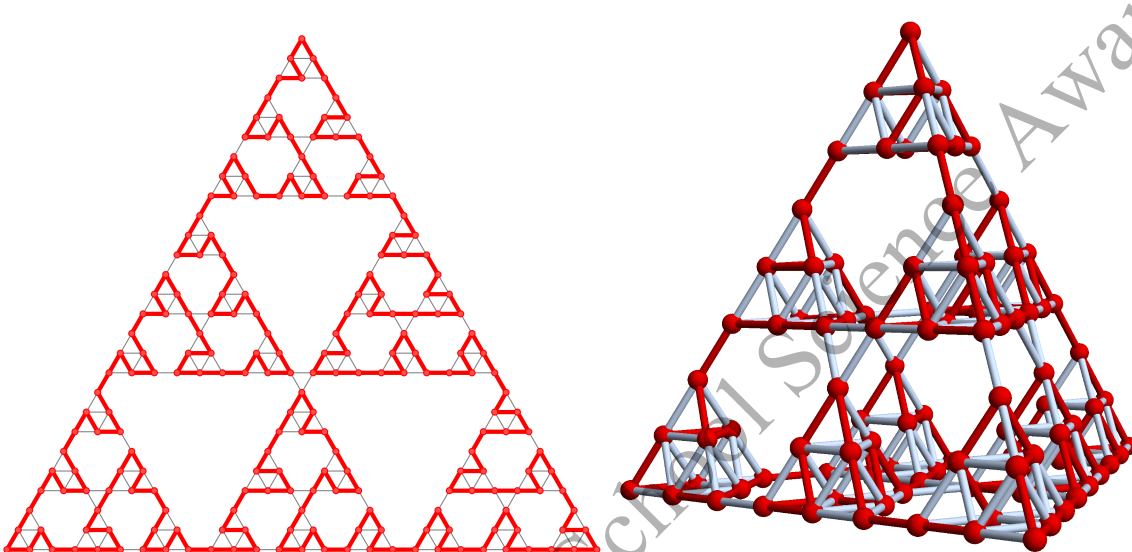


Figure 7: Funnel-free Hamiltonian cycles of A_3^3 and $A_{3_3}^2$.

8 A problem proposed by Teguia and Godbole

In this section, we apply results we obtained to prove the Hamiltonicity of the generalization of the Sierpiński gasket graphs proposed by Teguia and Godbole [20] by presenting a method to construct a Hamiltonian cycle of a generalized Sierpiński gasket graph from one of A_p^k. The proof is illustrated by Figure 8.

Definition 8.1. If $x = (x_0, x_1, x_2)$ is a vertex of $G(n)$, then $t(x)$ is the induced subgraph of $G(n+1)$ with vertices $\{(x_0, x_1, x_2 + 1), (x_0, x_1 + 1, x_2), (x_0 + 1, x_1, x_2)\}$.

Definition 8.2. If G is an induced subgraph of $G(n)$, then $T(G)$ denotes the graph $\cup_{v \in G} t(v)$.

Definition 8.3. The generalized Sierpiński gasket graph SG_p^k denotes $T(A_p^k)$.

Now SG_2^k is isomorphic to S_{k+1} in [20], and the desired generalization is SG_p^k where $p \geq 3$.

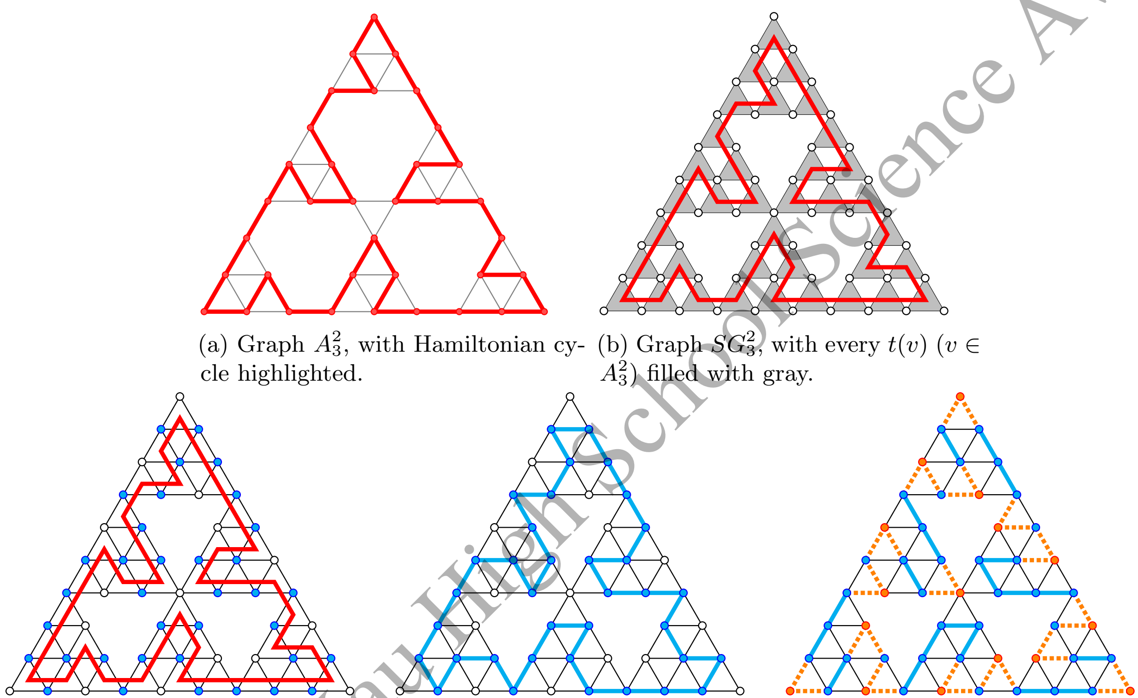
Lemma 8.1. *If u, v and w are different vertices in $G(n)$ and $t(v)$, $t(u)$ and $t(w)$ have a common vertex, then $\{u, v, w\}$ is a funnel.*

Proof. Let such a common vertex be (x_0, x_1, x_2). By definition, $u, v, w \in \{(x_0, x_1, x_2 - 1), (x_0, x_1 - 1, x_2), (x_0 - 1, x_1, x_2)\}$. Since u, v, w are pairwise different, $\{u, v, w\} = F((x_0, x_1, x_2))$. $\square$

Theorem 8.2. *SG_p^k is Hamiltonian.*

Proof. According to Theorem 7.8, there exists a funnel-free Hamiltonian cycle of A_p^k. Let such a cycle be $C = (v_1, v_2, \ldots, v_l)$. Let v_0 denote v_l and let v_{l+1} denote v_1. Let $u_{i,j}$ be the common vertex of $t(v_i)$ and $t(v_j)$. By Lemma 8.1, for any $1 \le i \le l$, $u_{i-1,i}$ and $u_{i,i+1}$ are different. So let w_i denote the vertex in $t(v_i)$ that is neither $u_{i-1,i}$ nor $u_{i,i+1}$.

Then we apply the following procedure to find a Hamiltonian cycle of SG_p^k. Let SC be the cycle $(u_{1,2}, \ldots, u_{l-1,l}, u_{l,1})$. We go from w_1 to w_l, and for each vertex w_i, if it is not already in SC, we insert it between $u_{i-1,i}$ and $u_{i,i+1}$. When we finish this process, SC will have become a Hamiltonian cycle of SG_p^k. □



(a) Graph A_3^2, with Hamiltonian cycle highlighted.

(b) Graph SG_3^2, with every $t(v)$ ($v \in A_3^2$) filled with gray.

(c) Graph SG_3^2, with vertices in SC (initial) highlighted.

(d) Graph SG_3^2, with SC (initial) highlighted.

(e) Graph SG_3^2, with SC (final) highlighted. Newly added edges and vertices are orange (the edges are also dotted) while old ones remain blue.

Figure 8: Illustration of the process generating a Hamiltonian cycle of SG_p^k.

9 Optimal pebbling of the triangular grid

The problem of graph pebbling was originally introduced to give a proof of a theorem in number theory [4]. Its history is outlined in two survey papers, [11] and [12], by Hurlbert. In [20], Teguia and Godbole asked about the pebbling number of S_n. They considered this problem to be "quite hard". We have not solved the problem, but we have made some progress. In this section, we give lower and upper bounds of the optimal pebbling number of the triangular grid graph. We begin by stating some important definitions about graph pebbling.

A *distribution* D of *pebbles* on a connected simple graph G is a function $D : V(G) \to \mathbb{N} \cup \{0\}$. A *pebbling move* from u to v removes two pebbles from u and add one pebble to v. A *pebbling sequence* is a sequence of pebbling moves. If $D(v) \geq 1$, or if there exists a pebbling sequence that places a pebble on v, then v is *reachable* from D. More generally, if $D(v) \geq k$, or if there exists a pebbling sequence that places k pebbles on v, then v is *k-reachable* from D. If every vertex of G is reachable from D, then D is *solvable*. The *pebbling number* $\pi(G)$ of a graph is the least number k such that any distribution with k pebbles is solvable. The *optimal pebbling number* $\pi_{\text{opt}}(G)$ of a graph is the least number k such that there exists a solvable distribution with k pebbles.

We will use a result in [8] to get a lower bound of the optimal pebbling number of the triangular grid graph. A few relevant definitions are helpful for the understanding of the method used.

The *coverage* of distribution D is the set of all vertices reachable from D. The *size of a distribution* D, denoted by $|D|$, is the total number of pebbles in D. The *size of the coverage* of a distribution D, denoted by $\text{Cov}(D)$, is the total number of vertices reachable from D. The *covering ratio* of a distribution D is given by $\frac{Cov(D)}{|D|}$. $\text{Reach}(D, v)$ is the greatest integer k such that v is k-reachable under D. The *excess* of v under D is denoted by $\text{Exc}(D, v)$, and is given by $\max(\text{Reach}(D, v) - 1, 0)$. The *total excess* of D is denoted by $\text{TE}(D)$, and is given by $\sum_{v\in V(G)} \text{Exc}(D, v)$. A *unit* is a distribution in which only one vertex contains pebbles. The *unit excess* of D is denoted by $\text{UE}(D)$, and is given by $\sum_{u|D(u)>0} TE(P_u)$, where P_u denotes a unit containing exactly $D(u)$ pebbles, all placed on u. The *effect* of a pebble placed at v is denoted by $\text{ef}(v)$, and is given by $\sum_{u\in V(G)} 2^{-d(v,u)}$.

Observe that if G is a spanning subgraph of G', then $\pi_{\text{opt}}(G) \geq \pi_{\text{opt}}(G')$. More generally, if $V(G')$ can be partitioned into n parts such that the subgraph induced by each part contains G as its spanning subgraph, then $\pi_{\text{opt}}(G) \geq \frac{\pi_{\text{opt}}(G')}{n}$. Therefore, we introduce the following graph.

Definition 9.1. The toroidal triangular grid graph is denoted by $TG(m, n)$. Its vertices are ordered pairs of integers (x_0, x_1) such that $0 \leq x_0 \leq m-1$, $0 \leq x_1 \leq n-1$. Two vertices (x_0, x_1) and (y_0, y_1) are adjacent if there exist integers $-1 \leq d_0, d_1 \leq 1$ such that $x_0 - y_0 \equiv d_0 \pmod m$, $x_1 - y_1 \equiv d_1 \pmod n$, and $|d_0| + |d_1| + |d_0 + d_1| = 2$.

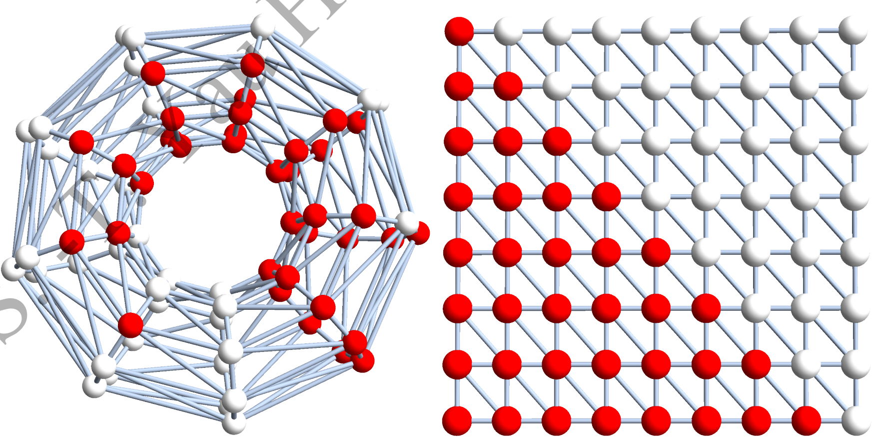


Figure 9: Toroidal triangular grid graph $TG(9, 8)$ and a spanning subgraph of it.

An example of a toroidal triangular grid graph is given by Figure 9. The vertex set of $TG(n+2, n+1)$ can be partitioned into two parts each of which induces $G(n)$, so a lower bound on the optimal pebbling number of toroidal triangular grid graphs naturally leads to one of triangular grid graphs. Since $TG(m,n)$ is vertex-transitive, we can apply a theorem in [8] to get a lower bound of its optimal pebbling number, namely $\frac{10}{107}mn$. We conjecture the lower bound to be $\frac{1}{5}mn$, which is 2.14 times the proven bound. For comparison, Győri et al. [8] applied the same theorem on the square grid $P_m \square P_n$ to get a lower bound $\frac{2}{13}mn$, but conjectured a lower bound of $\frac{2}{7}mn$ [7], which is $13/7 \approx 1.857$ times the proven bound. (Our bound is a strict inequality, and it is unclear why Győri et al. did not claim to prove a strict inequality when they actually did.)

Theorem 9.1. *For $m, n \geq 5$, the optimal pebbling number of $TG(m,n)$ is greater than $\frac{10}{107}mn$.*

Proof. As in [8], we make some claims regarding the coverage, excess and effect of units. We can see that when $m, n \geq 5$, for every unit U on $TG(m,n)$,

$$\mathrm{Cov}(U) \leq \begin{cases} 1 & |U| = 1 \\ \frac{7}{2}|U| & 2 \leq |U| \leq 3, \\ \frac{19}{4}|U| & |U| \geq 4 \end{cases} \text{ and } \mathrm{TE}(U) \geq \begin{cases} 0 & |U| = 1 \\ \frac{1}{2}|U| & 2 \leq |U| \leq 3, \\ 2|U| & |U| \geq 4 \end{cases}$$

and that for every vertex $v \in TG(m,n)$, $\mathrm{ef}(v) < 13$. Therefore, let P be an optimal distribution on $TG(m,n)$, and decompose P into disjoint unit distributions $U_1, U_2, \ldots, U_t$. Let S_1 be the total number of pebbles that are placed on units with size 1, and define $S_{2,3}$ and $S_{\geq 4}$ similarly. Then applying Corollary 5.2 of [8], we get

$$\begin{aligned} |P| &\geq \frac{\frac{\Delta-1}{\Delta-2}|V(TG(m,n))| + \mathrm{UE}(P) - \frac{1}{\Delta-2}\sum_{i=1}^{t}\mathrm{Cov}(U_i)}{\mathrm{ef}(v)} \\ &> \frac{\frac{5}{4}mn + \frac{1}{2}S_{2,3} + 2S_{\geq 4} - \frac{1}{4}(\frac{7}{2}S_{2,3} + \frac{19}{4}S_{\geq 4} + (|P| - S_{2,3} - S_{\geq 4}))}{13} \\ &= \frac{\frac{5}{4}mn - \frac{1}{8}S_{2,3} + \frac{17}{16}S_{\geq 4} - \frac{1}{4}|P|}{13} \\ &\geq \frac{\frac{5}{4}mn - \frac{1}{8}|P| - \frac{1}{4}|P|}{13} \end{aligned}$$

and therefore $|P| > \frac{10}{107}mn$. □

Corollary. *The optimal pebbling number of $G(n)$ is greater than $\frac{5(n+1)(n+2)}{107}$.*

Now we introduce a pebbling pattern that allows the covering ratio to be arbitrarily close to 5, which is about 83.3% the maximum degree. For comparison, we have mentioned that Győri et al. [7] conjectured the covering ratio for the square grid to be at most 3.5, which is 87.5% the maximum degree.

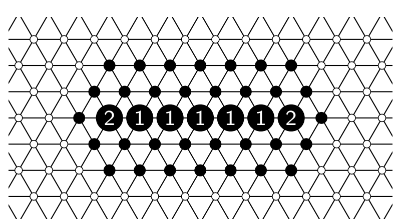


Figure 10: Elongating this structure increases its covering ratio.

Using a variant of this pattern, we obtain an upper bound for the optimal pebbling number of $G(n)$. The following lemmas that lead to the bound are illustrated by Figures 12 to 14.

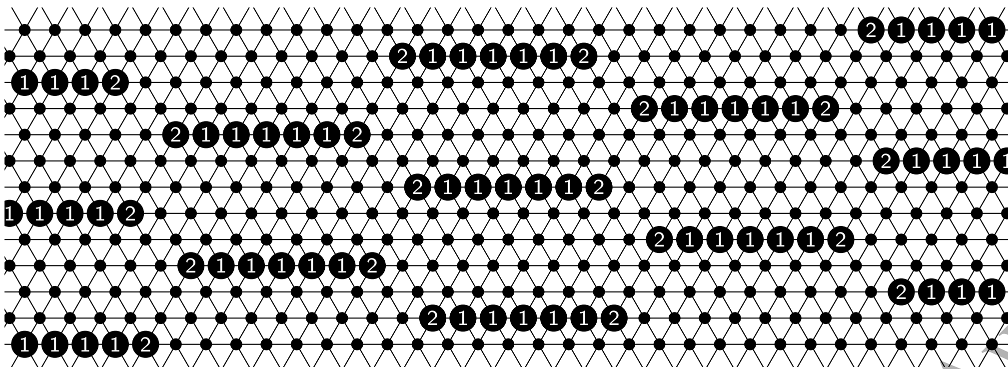


Figure 11: The above structure can be repeated to pebble infinitely large grids.

Lemma 9.2. *For* $0 \leq n \leq 6$, $\pi_{\text{opt}}(G(n)) \leq \max(n+1, 2n-2)$.

Lemma 9.3. *For* $7 \leq n \leq 14$, $\pi_{\text{opt}}(G(n)) \leq 3n-9$.

Lemma 9.4. *For* $n \geq 15$, $\pi_{\text{opt}}(G(n)) \leq 3n-9+\pi_{\text{opt}}(G(n-15))$.

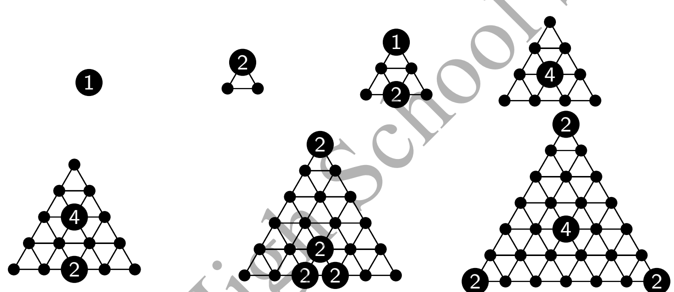


Figure 12: Proof of Lemma 9.2.

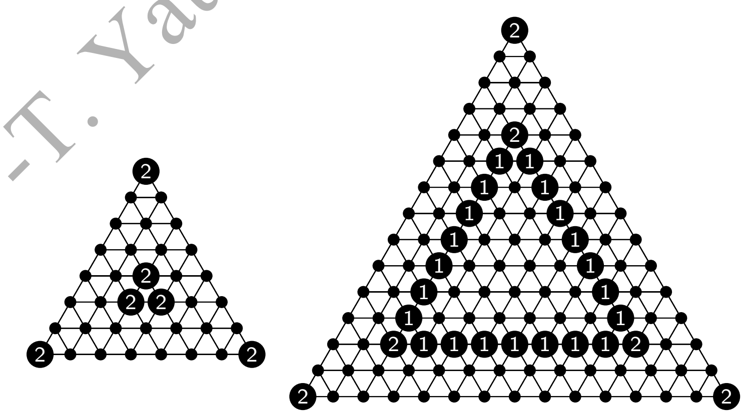


Figure 13: Proof of Lemma 9.3. The graphs in the figure are $G(7)$ and $G(14)$.

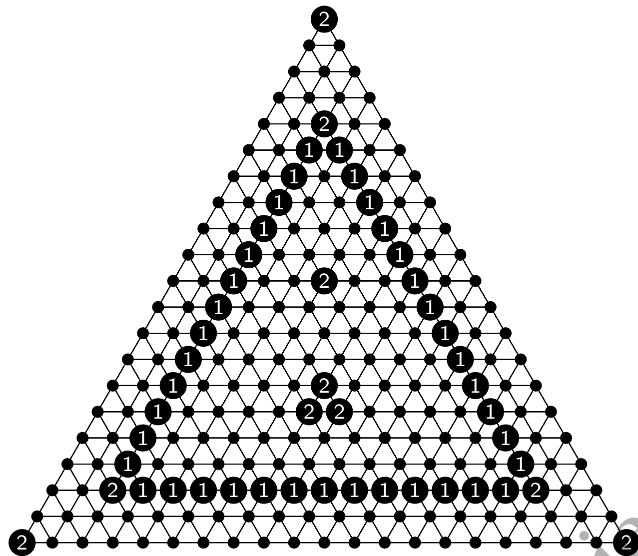


Figure 14: Proof of Lemma 9.4. The graph in the figure is $G(20)$. A part of this configuration is an optimal pebbling configuration of $G(5)$ that has been used in Figure 12.

Theorem 9.5. *The optimal pebbling number of $G(n)$ is at most $\frac{3}{2}d(15d+2r+9)+\max(r+1, 2r-2, 3r-9)$, where $n = 15d + r$, $0 \le r \le 14$.*

Proof. By Lemma 9.2 and Lemma 9.3, for $0 \le n \le 14$, $\pi_{\text{opt}}(G(n)) \le \max(n + 1, 2n - 2, 3n - 9)$, so this theorem is true for $d = 0$. Assume that it is true for some d. Then for any $n = 15(d + 1) + r$, $\pi_{\text{opt}}(G(n)) \le 3n-9+\pi_{\text{opt}}(G(n-15)) \le 3(15(d+1)+r)-9+\frac{3}{2}d(15d+2r+9)+\max(r+1, 2r-2, 3r-9) = \frac{3}{2}(d+1)(15(d+1)+2r+9)+\max(r+1, 2r-2, 3r-9)$, and the proof is complete. $\square$

We conjecture that this bound is asymptotically optimal. Indeed, we believe that a covering ratio equal to or greater than 5 is not achievable even on a toroidal triangular grid graph.

Conjecture. *For any positive integer m and n, the optimal pebbling number of $TG(m, n)$ is greater than $\frac{1}{5}mn$.*

10 Acknowledgements

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Appendix. Algorithm that finds Hamiltonian cycles of ${A_t}_p^k$

```
from numpy import array, concatenate, append, roll, where, argsort, arange
from math import comb
from numpy.linalg import norm
import matplotlib.pyplot as plt
from matplotlib.colors import LinearSegmentedColormap

def simplex_grid_path(t, n):
    '''
    Input: t, n

    Output: A numpy array describing a Hamiltonian path of
    the simplex grid graph G_t(n) with end vertices c_0 and c_t.

    Example:
    >>> simplex_grid_path(2,2)
    array([[2, 0, 0],
       [1, 0, 1],
       [1, 1, 0],
       [0, 2, 0],
       [0, 1, 1],
       [0, 0, 2]])
    '''
    if t==0:
        return array([[n]])

    return array([append([n-i], roll(v, n-i))
                  for i in range(n+1)
                  for v in simplex_grid_path(t-1, i)])

def Hamiltonian_cycle_of_A(t, p, k):
    '''
    Input: t, p, k

    Output: A numpy array describing a Hamiltonian cycle of
    the graph {A_t}_p^k.

    Example:
    >>> Hamiltonian_cycle_of_A(2, 2, 2)
    array([[2, 1, 0],
       [3, 0, 0],
       [2, 0, 1],
       [1, 0, 2],
       [0, 0, 3],
       [0, 1, 2],
       [0, 2, 1],
       [0, 3, 0],
       [1, 2, 0]])
    '''
    basic_path = simplex_grid_path(t, p-1)

    def get_basic_path(start, end):
        '''
        Returns a Hamiltonian path of G_t(p-1)
        starting at c_{start} and ending in c_{end}.
        '''
        inverse_permutation = [start] + list(set(range(t+1)) - {start, end}) + [end]
        permutation = argsort(inverse_permutation)
        return basic_path[:, permutation]
```

```
    component1 = simplex_grid_path(t, p-2)
    component1[:, [0, 1]] = component1[:, [1, 0]]
    component1 += array([1]+[0]*t)
    component2 = append(array([[0]]*comb(p+t-2, t-1)),
                        simplex_grid_path(t-1, p-1)[::-1],
                        axis=1)
    cycle = concatenate((component1, component2))

    for i in range(2, k+1):
        cycle = array([p * b + v
                       for a, b, c in zip(roll(cycle, 1, axis=0),
                                          cycle,
                                          roll(cycle, -1, axis=0))
                       for v in get_basic_path(where((a-b)==1)[0][0],
                                               where((c-b)==1)[0][0])])

    return cycle

plt.rcParams["figure.figsize"] = (8,4*(3**0.5))
position_matrix = [None,None,[[0,1,2],[0,3**0.5,0]],[[0,1,1,4],[0,1,4,1],[0,4,1,1]]]

def visualize_cycle(t, p, k):
    '''
    Input: t, p, k

    Output: Shows a figure showing a Hamiltonian cycle of {A_t}_p^k.
    No value is returned.

    When t=3, the interactive figure can be rotated and zoomed.

    It should be noted that 3D figures produced by matplotlib
    are subject to some glitches. For example, it is very often
    that an edge that should be under another edge is above it.
    However, matplotlib does have the virtue of being
    comparatively fast and interactive.
    '''
    if t==2:
        fig, ax = plt.subplots()
    elif t==3:
        fig = plt.figure(1)
        fig.add_subplot(111, projection='3d')
        ax = plt.axes(projection='3d')
    else:
        raise ValueError('Bad dimension %s.' % t)

    cycle = Hamiltonian_cycle_of_A(t, p, k)
    coord = array([sum((v*position_matrix[t]).transpose()).transpose() for v in cycle])
    length = len(cycle)

    for i in range(length):
        for j in range(i):
            if norm(cycle[i]-cycle[j], 1)!=2:
                continue
            ax.plot(*array([coord[i], coord[j]]).transpose(),
             c='red' if ((i-j-1)%length)*((j-i-1)%length)==0 else '#e0e0e0')
    ax.scatter(*coord.transpose(), c=arange(length), cmap="hsv", zorder=1000)

    plt.axis('off')
    plt.show()
```

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