Solving Megaminx puzzle With Group Theory

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First, this work proves the structure of the megaminx group: semidirect product of a orientation twisting subgroup and a position permutation subgroup, the former subgroup is decomposed further into the product of multiplicative groups of integers modulo 2 or 3, the later subgroup is the product of alternating groups. Second, the work gives the sufficient and necessary conditions for a configuration to be solvable. Third, the work shows a constructive algorithm to solve the megaminx, which is suitable for the blindfolded competition.

Contributions

1. prove the structure of the megaminx move group, theorem 3.12;
2. give sufficient and necessary conditions for a configuration to be solvable, theorem 3.8;
3. construct an algorithm to solve megaminx, corollary 3.4.

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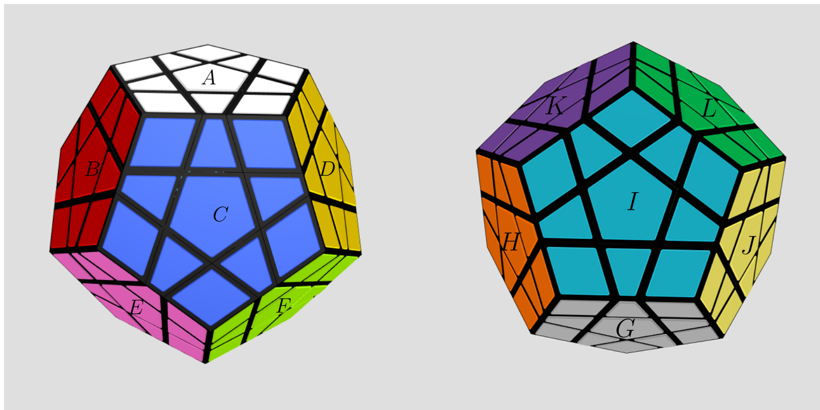
Keywords: Group, Semidirect product, Commutator, Conjugate, Generators

1. Introduction

The Rubik's cube is a family of combination puzzles that has become very popular in various cultures since its invention in 1974 by Ernö Rubik. The mathematical structure underlying the Rubik's cube is both profound and fascinating, it has intrinsic connections with group theory [1]. The classical Rubik's cube has been generalized to many different types of puzzles, such as pyraminx, megaminx and so on. Comparing to the classical Rubik's cube, megaminx has much more complicated combinatorial structures but is much less studied. Although there are several manual receipts to solve megaminx, there are few works to clarify the megamnix group structure and the solvability in a rigorous mathematical way. This project aims to describe the structure of the group of all megaminx moves, clarify the sufficient and necessary condition for a solvable state, and construct a sequence of moves to solve a given state.

1.1. Rubik's Cube Group

The Rubik's cube has been thoroughly studied using group theory, lecture notes can be found in [2] and [3]. Suppose $\mathbb{G}$ represents all the moves of a Rubik's cube, $*$ is the composition operator of the moves, then $(\mathbb{G}, *)$ forms a non-Abelian group. Let G_O be the set of moves which fix the positions of all the cubies but permute the orientations, then G_O is a normal subgroup of $\mathbb{G}$. Let G_P be the set of moves with preserve the orientations of all the cubies but permuate the positions, then G_P is a



(a). front view (b). back view

Figure 1: A megaminx with face labels, the megaminx is a dodecahedron, the 12 faces are labeled as $\{A,B,C,D,E,F,G,H,I,J,K,L\}$. Each face has one opposite face, there are 6 pairs in total, $\{A,G\}$, $\{B,H\}$, $\{C,I\}$, $\{D,J\}$, $\{E,K\}$ and $\{F,L\}$.

non-normal subgroup. $\mathbb{G}$ is the semidirect product of G_O and G_P, $\mathbb{G} = G_O \rtimes G_P$. Furthermore, the total orientation of 8 edge cubies is even, the total orientation of 12 corner cubies is divisible by 3, hence $G_O = \mathbb{Z}_2^{11} \times \mathbb{Z}_3^7$. The parities of the permuation of the corner cubies and that of the edge cubies are equal, therefore $G_P = (A_{12} \times A_8) \rtimes \mathbb{Z}_2$. Therefore, the Rubik's cube group has the structure

$$\mathbb{G} = G_O \rtimes G_P = (\mathbb{Z}_2^{11} \times \mathbb{Z}_3^7) \rtimes ((A_{12} \times A_8) \rtimes \mathbb{Z}_2).$$

1.2. Rubi's Cube Algorithms

There are three types of algorithms to solve a Rubik's cube, all of them are based on group theory and tailored for different type of competitions.

In the first type, a human player solves a Rubik's cube by memorizing a set of rules and special patterns of configurations, and by observing the current pattern to select a rule to further transform, such as the most popular seven step method [4];

The second type of algorithms are for blindfolded competition, a human player observes and memorizes the starting configuration in mind, and perform the moves without further observations. The player needs to memorize a few generators of $\mathbb{G}$ and use conjugation trick to generalize them for all situations. More specifically, the generators of G_P are corner cubie 3-cycles and edge cubie 3-cycles, the generators of G_O are pair corner cubie twists and pair edge cube flips, which will be explained in details in subsection 3.3.

The third type of algorithms are performed by computers. Thistlethwaite's algorithm [5] finds a sequence of nesting normal subgroups

$$\langle e \rangle = G_4 \trianglelefteq G_3 \trianglelefteq G_2 \trianglelefteq G_1 \trianglelefteq G_0 = \mathbb{G},$$

each G_k acts on the solved configuration to generate a space S_k, then we obtain a sequence of nesting configuration spaces

$$S_4 \subset S_3 \subset S_2 \subset S_1 \subset S_0,$$

where S_4 contains only the solved configuration, S_0 is the space of all possible solvable configurations. At each step, the Thistlethwaite algorithm finds a move in G_{k-1}/G_k to move the current configuration from S_k to S_{k-1}. The upper bound is 52. Kunkle and Cooperman improved the bound to 26 in [6]. In 2010, the sharp upper bound of steps to solve a Rubik's cube, namely the God's number, is proven to be 20 with the aid of huge computational resources [7].

1.3. Contributions

To the best of the knowledges of the author, the God's number for Megaminx is widely open today. The group structure of Megaminx, the type two and three algorithms haven't been systematically studied. The current project focuses on studying the group structure and the second type of computational algorithm of Megaminx. Therefore, our main contributions are

- prove the structure of the megaminx move group, theorem 3.12;
- give sufficient and necessary conditions for a configuration to be solvable, theorem 3.8;
- construct an algorithm to solve megaminx, corollary 3.4.

2. Basic Concepts in Group Theory

This section reviews some basic concepts in group theory, and facts for symmetry group, then introduces the techniques of commutator and conjugation to generate 3-cycles.

2.1. Basic Concepts

Definition 2.1 (Group)**.** *A group $(G,*)$ consists of a set $\mathbb{G}$ and an operation $*$ such that:*

1. *$\mathbb{G}$ is closed under $*$, for all $a,b \in G$, then $ab \in G$;*
2. *$*$ is associative, for all $a,b,c \in G$, $(a*b)*c = a*(b*c)$;*
3. *there is an identity element $e \in G$, which satisfies $e*g = g*e = g$, for all $g \in G$;*
4. *inverse exists, for any $g \in G$, there exists an $h \in G$, such that $g*h = h*g = e$.*

Definition 2.2 (Subgroup)**.** *A nonempty set H of a group $(G,*)$ is called a subgroup of G if $(H,*)$ is a group.*

Definition 2.3 (Generator)**.** *Let G be a group and S be a subset of G. We say S is a set of generators of G, if $G = \langle S \rangle$; that is, every element of G can be written as a finite product (under the group operation) of elements of S and their inverse.*

Definition 2.4 (Normal Subgroup)**.** *A subgroup N of G is said to be a normal subgroup, if*

$$g^{-1}Ng \subseteq N \quad \forall g \in G,$$

and denoted as $N \trianglelefteq G$*.*

Definition 2.5 (Quotient Group)**.** *Let N be a normal subgroup of a group G. We define the set G/N to be the set of all left cosets of N in G,*

$$G/N = \{gN : g \in G\}.$$

Define an operation on G/N as follows. For each aN and bN in G/N, the product

$$(aN) \cdot (bN) = (ab)N.$$

Then $(G/N, \cdot)$ *forms a group, the so-called quotient group.*

Definition 2.6 (Direct Product)**.** *Given groups* $(G, *)$ *and* $(H, \star)$*, the direct product* $G \times H$ *is defined as follows:*

1. *The underlying set is the Cartesian product, GH. That is, the ordered pairs* (g, h)*, where* $g \in G$ *and* $h \in H$*.*
2. *The binary operation on* $G \times H$ *is defined component-wise:* $(g_1, h_1) \cdot (g_2, h_2) = (g_1 * g_2, h_1 \star h_2)$

then $(G \times H, \cdot)$ *forms a group.*

Let G be a group, H and K are subgroups satisfying: H and K are normal in G, $H \cap K = \{e\}$, $HK = G$, then G is isomorphic to the direct product $H \times K$.

Definition 2.7 (Semidirect Product)**.** *Let G be a group, H and K are subgroups satisfying:*

1. *H is normal in G,*
2. $H \cap K = \{e_G\}$*,*
3. $HK = \{hk | h \in H, g \in K\} = G$*,*

then G is the semidirect product of H and K, denoted as $G = H \rtimes K$*.*

If the group G is a semi-direct product of its subgroups H and Q, then the semi-direct Q is isomorphic to the quotient group G/H.

Definition 2.8 (Homomorphism)**.** *Let* $(G, *)$ *and* $(H, \star)$ *be two groups. A homomorphism from G to H is a map* $\varphi : G \to H$ *such that* $\varphi(a * b) = \varphi(a) \star \varphi(b)$ *for all* $a, b \in G$*.*

It can be shown that the image of φ, *im* $\varphi = \{\varphi(g) : g \in G\}$, is a subgroup of H.

Definition 2.9 (Kernel)**.** *The kernel of a homomorphism* $\varphi : G \to H$ *is defined to be* $\{g \in G : \varphi(g) = e_H\}$*. That is, ker* φ *is the pre-image of* e_H *in G.*

It can be shown that the kernel of φ, *ker* φ, is a subgroup of G.

Definition 2.10 (Right Group Action)**.** *A right group action of a group* $(G,*)$ *on a non-empty set A is a map* $A\times G\to A$ *satisfying the following properties:*

1. $(a\cdot g_1)\cdot g_2 = a\cdot(g_1 * g_2)$ *for all* $g_1,g_2\in G$ *and* $a\in A$*.*
2. $a\cdot e = a$ *for* $a\in A$*.*

Definition 2.11 (Transitive Action)**.** *If a group G acts on a set A, then the orbit of* $a\in A$ *is the set* $\{a\cdot g : g\in G\}$*. If a group action has only one orbit, we say that the action is transitive.*

G acts on the set of ordered pairs (C_1,C_2) of different unoriented corner(edge) cubies, $(C_1,C_2)\cdot M = (M(C_1),M(C_2))$. This action is transitive. In the same way, G acts on the set of ordered triples (C_1,C_2,C_3) of different unoriented corner(edge) cubies.

2.2. Symmetry Group

Definition 2.12 (Symmetry Group)**.** *The symmetric group on n letters is the set of bijections from* $\{1,2,\dots,n\}$ *to* $\{1,2,\dots,n\}$*, with the operation of composition, and denoted as* S_n*.*

Definition 2.13 (Cycle)**.** *The k-cycle* $(i_1 i_2\cdots i_k)$ *is the element* $\tau\in S_n$*, defined by*

$$\tau(i_1)=i_2,\tau(i_2)=i_3,\cdots,\tau(i_{k-1})=i_k,\tau(i_k)=i_1,$$

and $\tau(j)=j$ *if* $j\neq i_r$ *for any r. The support of the cycle is the set* $\{i_1,i_2,\dots,i_k\}$ *of numbers which appear in the cycle, and denoted as* $\mathrm{supp}\,\tau$*.*

Two cycles τ and σ are disjoint, if they have no numbers in common, namely $\mathrm{supp}\,\sigma\cap\mathrm{supp}\,\tau=\emptyset$. If $\sigma,\tau\in S_n$ are disjoint cycles, then $\sigma\tau=\tau\sigma$. Any $\sigma\in S_n$ can be written as a product of disjoint cycles, this product is called the disjoint cycle decomposition of σ.

S_n is generated by the 2-cycles, namely, any permutation in S_n can be written as a finite product of 2-cycles. If a permutation $\sigma\in S_n$ is a product of an even number of 2-cycles, then σ is called even; if σ is a product of an odd number of 2-cycles, then σ is called odd.

Definition 2.14 (Alternating Group)**.** *All the even permutations in* S_n *form a subgroup of* S_n*, which is called the alternating group, and denoted as* A_n*.*

Alternating groups are generated by 3-cycles.

2.3. Commutator and Conjugate

Definition 2.15 (Commutator)**.** *Suppose* $\sigma,\tau\in G$*, the commutator of* σ *and* τ $[\sigma,\tau]$ *is defined as*

$$[\sigma,\tau]=\sigma\tau\sigma^{-1}\tau^{-1}.$$

If σ and τ have disjoint support, namely no overlap, then they commute, $[\sigma,\tau]=e$; if their supports have only a small amount of overlap, then σ and τ almost commute.

Lemma 2.1 (Commutator)**.** *Suppose $\sigma, \tau \in S_n$, and $supp(\sigma) \cap supp(\tau) = \{y\}$, $\sigma(x) = y$ and $\tau(y) = z$, then $[\sigma, \tau]$ is at most a 3-cycle (x, y, z).*

Proof. The proof is based on direct computation,

$$\begin{aligned}
[\sigma,\tau](x) &= \sigma\tau\sigma^{-1}\tau^{-1}(x) = \tau\sigma^{-1}\tau^{-1}(\sigma(x)) = \tau\sigma^{-1}\tau^{-1}(y)\\
&= \sigma^{-1}\tau^{-1}(\tau(y)) = \tau^{-1}(\sigma(\tau(y))) = \tau^{-1}(\tau(y)) = y.\\
[\sigma,\tau](y) &= \sigma\tau\sigma^{-1}\tau^{-1}(y) = \tau\sigma^{-1}\tau^{-1}(\sigma(y)) = \sigma^{-1}\tau^{-1}(\tau(\sigma(y)))\\
&= \sigma^{-1}\tau^{-1}(\sigma(y)) = \tau^{-1}(\sigma^{-1}(\sigma(y))) = \tau^{-1}(y) = z.\\
[\sigma,\tau](z) &= \sigma\tau\sigma^{-1}\tau^{-1}(z) = \tau\sigma^{-1}\tau^{-1}(\sigma(z)) = \tau\sigma^{-1}\tau^{-1}(z) = \sigma^{-1}\tau^{-1}(\tau(z))\\
&= \sigma^{-1}\tau^{-1}(y) = \tau^{-1}(\sigma^{-1}(y)) = \tau^{-1}(x) = x
\end{aligned}$$

For any other element $\gamma \notin \{x, y, z\}$, then it is straight forward to show $[\sigma, \tau](\gamma) = \gamma$. □

Definition 2.16 (Conjugate)**.** *Let G be a group. Two elements σ and τ of G are conjugate, if there exists an element γ in G such that*

$$\gamma\sigma\gamma^{-1} = \tau.$$

One says also that τ is a conjugate of σ and that σ is a conjugate of τ.

Lemma 2.2 (Conjugate)**.** *Suppose $\sigma \in S_n$ is a cycle $\sigma = (i_1 i_2 \ldots i_k)$, and $\gamma \in G$, such that $\gamma(j_l) = i_l$, $l = 1, 2, \ldots, k$, then*

$$\gamma\sigma\gamma^{-1} = (j_1 j_2 \ldots j_k).$$

Proof. The proof is based on direct computation,

$$\gamma\sigma\gamma^{-1}(j_l) = \sigma\gamma^{-1}(\gamma(j_l)) = \sigma\gamma^{-1}(i_l) = \gamma^{-1}(\sigma(i_l)) = \gamma^{-1}(i_{l+1}) = j_{l+1}.$$

For any other element $x \notin \{i_1, i_2, \ldots, i_k\}$, then it is straight forward to show $\gamma\sigma\gamma^{-1}(x) = x$. □

3. Megaminx Group

This section studies the megaminx group structure, shows the sufficient and necessary conditions for solvable configurations and construct an algorithm to solve megaminx.

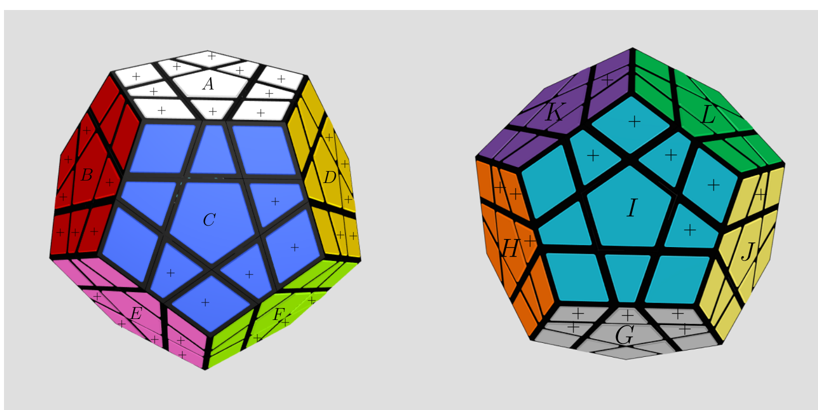
3.1. Megaminx notations

The megaminx is a dodectahedron, with 12 faces, each face is a pentagon. The faces are labeled using capital letters from A to L, as shown in Fig. 1.

Cubies. The megaminx is composed of 62 small cubes, which are typically called "cubies". The cubies in corners are called "corner cubies". Each corner cubie has 3 visible faces, and there are 20 corner cubies. The cubies with 2 visible faces are called "edge cubies", there are 30 edge cubies. The cubes with a single visible face are called "center cubies". There are 12 center cubies.

To name a corner cubie, we simply list its visible faces in counter-clockwise order using lower case letters. For example, the corner cubie at the intersection of three faces A, B, C is denoted as abc. If we don't consider the orientation, we can also call the cubie as bca and cab. In this situation, the cubie is called "unoriented cubie". If we care about which face of the cubie is listed first, then the cubie is called "oriented cubie". The oriented cubies abc and bca are different but the unoriented cubies, abc and bca are the same. Similarly, to name edge and center cubies, we just list the visible faces of the cubies. We also frequently talk about "cubicles", which are labeled the same way as cubies, but they describe the space in which the cubie lives. The cubicles are labeled the same way as cubies. When we move the megaminx, the cubies are moved, but the cubicles remain unchanged. However, we assume when we rotate a face of the megaminx, all center cubies stay in their cubicles, for the purposes to remove some global symmetry.

Moves. Furthermore, we need to name the moves of the megaminx. We use a capital letter A to represent a rotation of face A, the rotation is counter-clockwise with respect to the normal to the face A. A^{-1} represents a clockwise rotation of face A with respect to the normal. Similarly, each letter from A to L represents a counter clock-wise rotation of the corresponding face. These rotations are called basic moves.



(a). front view (b). back view

Figure 2: The cross field is defined on the megaminx in order to define the orientation of all the cubies.

Orientations. The orientation of the cubies are defined in a complicated way. For each unoriented corner cubie with a label $c_1c_2c_3$, if c_k is the smallest in the lexicographical order of letters, $k = 1, 2, 3$, then we label a cross on the c_k face of the cubie. Similarly, for each edge cubie with a label c_1c_2, we put a cross on the face of the cubie with the smaller letter. Each cubie has a unique face with a cross, which we call as primary face. Suppose after some moves, an oriented corner cubie reaches the position of a cubicle. Suppose the primary face of the cubie coincides with the primary face of the cubicle, then the orientation of the cubie is 0; if we need to rotate the cubie counter-clockwisely by an angle $\frac{2\pi}{3}$ to align the two primary faces, then the orientation of the cubie is 1; if we need to rotate the cubie counter-clockwisely by an angle $\frac{4\pi}{3}$ to align the two primary faces, then the orientation of the cubie is 2. We can define the orientation of an edge cubie in the similary way. Suppose the primary face of the cubie coincides with the primary face of the cubicle, then the orientation of the cubie is 0; if we need to rotate the cubie counter-clockwisely by an angle π to align the two primary faces, then the orientation of the cubie is 1.

Configurations. For each corner cubie, we simply list its visible faces in counter-clockwise order using lower case letters, the letter on the primary face is at the first. Then we sort all the corner cubies using lexicographical order of their names, and denoted as $\{C_1, C_2, \cdots, C_{20}\}$. The edge cubies are sorted in the similar way, and denoted as $\{E_1, E_2, \cdots, E_{30}\}$. In each configuration, the permutation of the unoriented corner cubies is denoted as σ, the permutation of the unoriented edge cubies is denoted as τ. The orientations of the corner cubies are represented as a vector $\mathbf{x} = (x_1, x_2, \cdots, x_{20})$, $x_k \in \mathbb{Z}_3$, $k = 1, 2, \ldots, 20$, representing the orientation of C_k. The orientations of all edge cubies are represented as a vector $\mathbf{y} = (y_1, y_2, \cdots, y_{30})$, $y_i \in \mathbb{Z}_2$, $i = 1, 2, \ldots, 30$, representing the orientation of E_i. The whole configuration is represented by $(\sigma, \tau, \mathbf{x}, \mathbf{y})$. The initial solved configuration is denoted as $(\sigma_0, \tau_0, \mathbf{0}, \mathbf{0})$.

3.2. Megaminx Group Definition and Properties

Definition 3.1 (Megaminx Move)**.** *A megaminx move is the rotation of a particular face of the dodecahedron in the counter-clockwise direction by* $\frac{2\pi}{5}$.

We refer each move using the same capital letter of the corresponding dodecahedron face. We write the set of basic moves as

$$\{A, B, C, D, E, F, G, H, I, H, K, L\} \tag{1}$$

where the move A rotate the dodecahedron face with label A by $\frac{2\pi}{5}$ counter-clockwisely. The other moves are denoted in the same way. We can make the set of moves of the megaminx into a group, denoted as $(\mathbb{G}, *)$. The elements of $\mathbb{G}$ are all possible moves of the megaminx. Two moves are considered the same if they result in the same configuration of the megaminx. If M_1 and M_2 are two moves, then $M_1 * M_2$ is the move where we first perform M_1 and then do M_2.

If C is an oriented cubie, we write $M(C)$ for the oriented cubicle that C ends up in after applying the move M, with the faces of $M(C)$ written in the same order as the faces of C. In more details, the first face of C should end up in the first face of $M(C)$, and so on.

Theorem 3.1. *The set of megaminx moves forms a group* $(\mathbb{G}, *)$*, with the operation* $*$ *on moves being the composition of corresponding sequence of moves. We will call this* *group* $(\mathbb{G},*)$ *the Megamnix Group, with generators*

$$\mathbb{G} = \langle A, B, C \cdots, K, L \rangle.$$

Proof. The proof is straight forward by using the definition of group.

- $\mathbb{G}$ is certainly closed under the operation $*$, since if M_1 and M_2 are moves, $M_1 * M_2$ is a move as well.
- If we let e be the empty move, then $M*e$ means first perform M and then do nothing, which is the same as just doing M, so $M*e = M$. $(\mathbb{G},*)$ has an identity.
- Suppose M is a basic move, then $M^5 = e$, namely $M^{-1} = M^4$. For compound move
$$(M_1 M_2 \ldots M_k)^{-1} = M_k^{-1} M_{k-1}^{-1} \ldots M_1^{-1},$$
where M_i's are basic moves.
- The operation $*$ is associative. A move can be defined by the change in configuration it causes, the associativity can be easily verified as follows: first, we investigate what a sequence of two moves does to the megaminx. If M_1 and M_2 are two moves, M_1 moves C to the cubicle $M_1(C)$, the move M_2 moves it to $M_2(M_1(C))$, therefore $(M_1 * M_2)(C) = M_2(M_1(C))$. Second, consider a sequence of three moves,
$$[(M_1 * M_2) * M_3](C) = M_3([M_1 * M_2](C)) = M_3(M2(M1(C))),$$
on the other hand
$$[M_1 * (M_2 * M_3)](C) = (M_2 * M_3)(M_1(C)) = M_3(M_2(M_1(C))).$$
Therefore $(M_1 * M_2) * M_3 = M_1 * (M_2 * M_3)$.

Therefore, $(\mathbb{G},*)$ is indeed a group. □

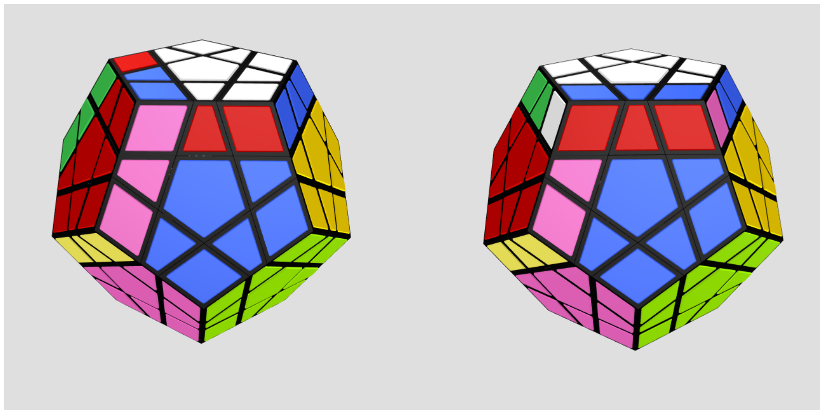
We can write each move of the megaminx using a modified cycle notation, which describes *where* each oriented cubie moves and *where* each face of the cubie moves. For example,

$$A = (abc, acd, adk, akl, alb)(ac, ad, ak, al, ab).$$

Proposition 3.2. *The megaminx group* $(\mathbb{G},*)$ *is non-Abelian.*

Proof. As shown in Fig. 3, the move $A^{-1}B^{-1}$ is not equal to $B^{-1}A^{-1}$. Therefore $\mathbb{G}$ is non-Abelian. □

We can define a map $\varphi_c : \mathbb{G} \to A_{20}$ as follows: any move M in $\mathbb{G}$ rearranges the corner cubies, it defines a permutation of the 20 unoriented corner cubies. Furthermore, all the basic moves in Eqn. 1 produce a 5-cycle, therefore all the corner cubie permutations are even. Hence $\varphi_c(\mathbb{G}) \subseteq A_{20}$. Similarly, we define $\varphi_e : \mathbb{G} \to A_{30}$: any move M reduces an even permuation of 30 unoriented edge cubies, $\varphi_e(\mathbb{G}) \subseteq A_{30}$.



(a). $A^{-1}B^{-1}$ (b). $B^{-1}A^{-1}$

Figure 3: The move AB is not equal to BA, the megaminx gorup $\mathbb{G}$ is non-Abelian.

3.3. Generators Construction

Lemma 3.3. *The action of the megaminx group on unoriented cubies are transitive, more specifically*

1. *The group $\varphi_c(\mathbb{G})$ acts on the set of ordered pairs (C_1, C_2) of different unoriented corner cubies is transitive;*
2. *The group $\varphi_c(\mathbb{G})$ acts on the set of ordered triples (C_1, C_2, C_3) of different unoriented corner cubies is transitive;*
3. *The group $\varphi_e(\mathbb{G})$ acts on the set of ordered pairs (C_1, C_2) of different unoriented edge cubies is transitive;*
4. *The group $\varphi_e(\mathbb{G})$ acts on the set of ordered triples (C_1, C_2, C_3) of different unoriented edge cubies is transitive.*

Proof. The proof is straightforward. We focus on 1 only, the other proofs are similar. Given an arbitrary ordered unoriented corner cubies (C_1, C_2), it is necessary and sufficient to show that they can be transformed to (abc, acd). There are 3 major steps,

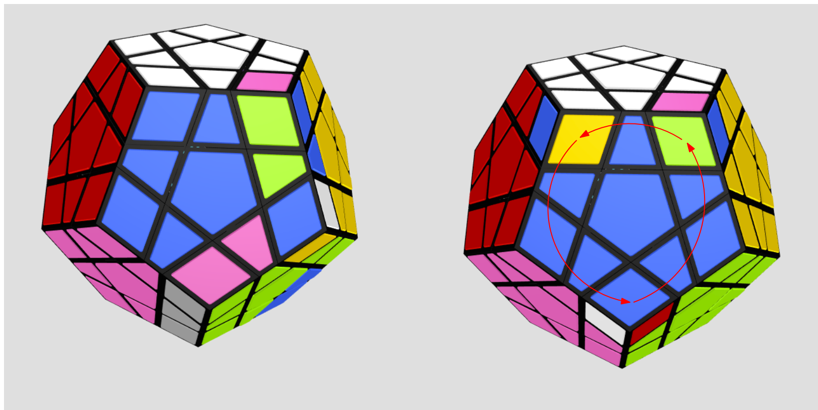
1. move both C_1 and C_2 to face C,
2. rotate face C to move C_1 to the cubicle abc,
3. use E, F and D or their inverses to move C_2 to acd.

□

Lemma 3.4 (Corner 3-cycle)**.** *There is a corner 3-cycle move in $M \in \mathbb{G}$, $M = (C_1, C_2, C_3)$ where C_k's are unoriented corner cubies.*

Proof. As shown in Fig. 4, we construct a corner 3-cycle using the commutator trick as described in Lem 2.1. Let $\sigma = DF^{-1}D^{-1}$ and $\tau = A$, then the support of σ and the support of τ has one corner cubie acd, hence the commutator is the 3-cycle

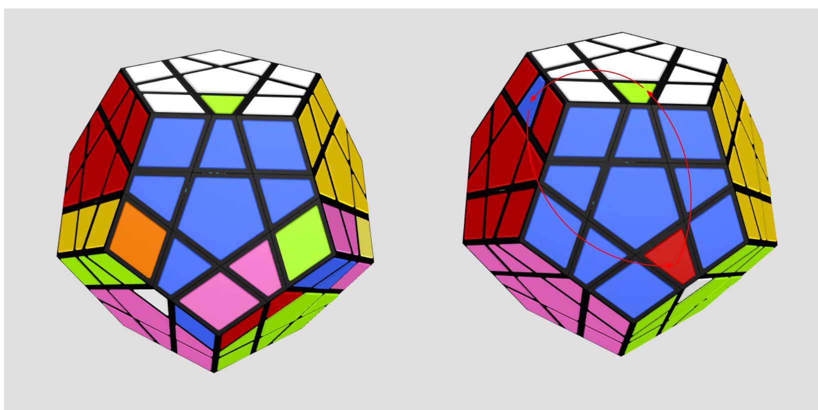
$$[\sigma, \tau] = (abc, cef, acd).$$



(a). $\sigma = DF^{-1}D^{-1}$ (b). corner 3-cycle

Figure 4: The 3-cycle of unoriented corner cubies.

$\square$



(a). $\sigma = BD^{-1}C^2DB^{-1}$ (b). edge 3-cycle

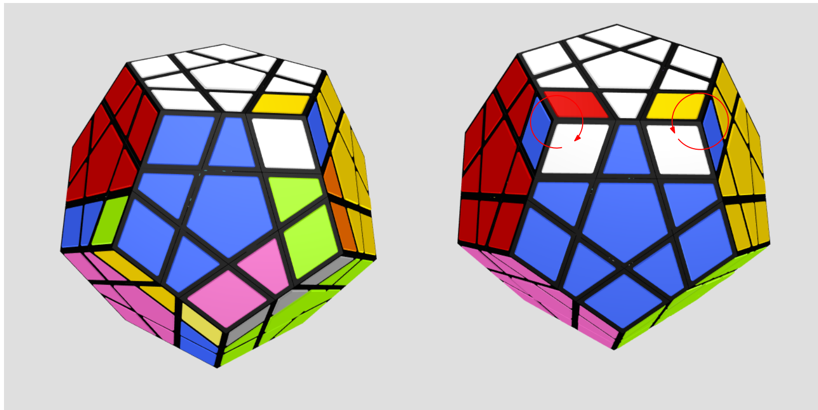
Figure 5: The 3-cycle of unoriented edge cubies.

Lemma 3.5 (Edge 3-cycle)**.** *There is a edge 3-cycle move in $M \in \mathbb{G}$, $M = (C_1, C_2, C_3)$ where C_k's are unoriented edge cubies.*

Proof. As shown in Fig. 5, we construct an edge 3-cycle using the commutator trick as described in Lem 2.1. Let $\sigma = BD^{-1}C^2DB^{-1}$ and $\tau = A$, then the support of σ and the support of τ has one edge cubie ac, hence the commutator is the 3-cycle

$$[\sigma, \tau] = (ac, ab, cf).$$

$\square$



(a). $\sigma = DF^2EF^{-1}D^{-1}$ (b). corner 2-twist

Figure 6: The twists of a pair of oriented corner cubies.

Lemma 3.6 (Corner 2-twist)**.** *There is a corner 2-twist move in $M \in \mathbb{G}$, M preserves the positions of both corner and edge unoriented cubies, but twists a pair of corner cubies (C_1, C_2), such that the orientation of C_1 increases by $+1$, and the orientation of C_2 decreases by -1.*

Proof. As shown in Fig. 6, we construct a corner twist move using the commutator trick as described in Lem 2.1. Let $\sigma = DF^2EF^{-1}D^{-1}$ and $\tau = A$. The support of $\varphi_c(\sigma)$ and the support of $\varphi_c(\tau)$ has no overlap, hence the commutator $[\sigma, \tau]$ is the identity on the unoriented corner cubies.

But the commutator twists the orientations of two corner cubies, *acd* is twisted by $+1$, *abc* is twisted by -1,

$$[\sigma, \tau] = (abc, cab)(acd, cda) \tag{2}$$

□

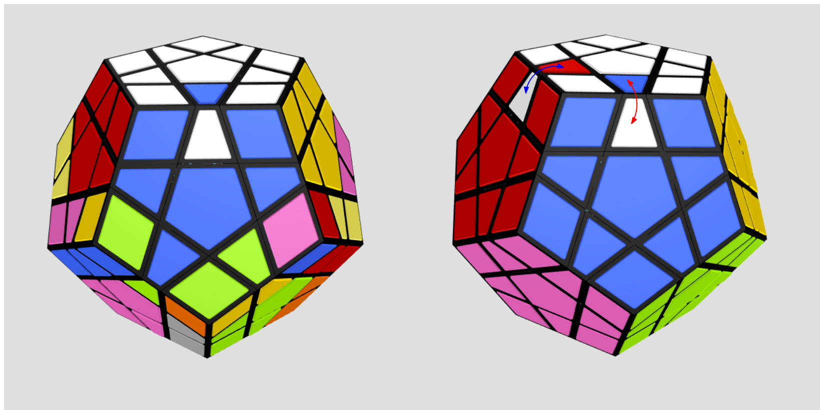
Lemma 3.7 (Edge 2-Flip)**.** *There is a edge 2-flip move in $M \in \mathbb{G}$, M preserves the positions of both corner and edge unoriented cubies, but twists a pair of edge cubies (C_1, C_2), such that the orientation of C_1 increases by $+1$, and the orientation of C_2 decreases by -1.*

Proof. As shown in Fig. 7, we construct an edge flip move using the commutator trick as described in Lem 2.1. Let $\sigma = \sigma = BD^{-1}C^2E^{-1}F^{-1}C^2DB^{-1}$ and $\tau = A$. The support of $\varphi_e(\sigma)$ and the support of $\varphi_e(\tau)$ has no overlap, hence the commutator $[\sigma, \tau]$ is the identity on the unoriented edge cubies.

But the commutator flips the orientations of two edge cubies, *ab* is flipped, *ac* is flipped,

$$[\sigma, \tau] = (ab, ba)(ac, ca) \tag{3}$$

□



(a). $\sigma = BD^{-1}C^2E^{-1}F^{-1}C^2DB^{-1}$ (b). edge 2-flip

Figure 7: The flips of a pair of oriented edge cubies.

3.4. Solvable Configuration

Definition 3.2 (Solvable Configuration)**.** *Suppose* $(\sigma, \tau, \mathbf{x}, \mathbf{y})$ *is a configuration, if there is a move g in* $\mathbb{G}$*, such that g acts on the solved configuration will result in the* $(\sigma, \tau, \mathbf{x}, \mathbf{y})$*, then this configuration is called solvable.*

Theorem 3.8 (Solvable Configuration)**.** *A* $(\sigma, \tau, \mathbf{x}, \mathbf{y})$ *is solvable if and only if*

1. *the parity of* σ *equals to the parity of* σ_0*;*
2. *the parity of* τ *equals to the parity of* τ_0*;*
3. *the orientations satisfy*

$$\sum_{i=1}^{20} x_i \equiv 0 \; (mod\ 3), \qquad \sum_{j=1}^{30} y_j \equiv 0 \; (mod\ 2).$$

Proof. Necessary condition. All the basic moves in Eqn. 1 induce 5-cycles of corner cubies and edge cubies, which are even permuations. Hence the pairity of σ and τ must equal to those of σ_0 and τ_0. This shows the first 2 conditions hold.

By direct verification, under the basic moves, condition 3 holds, namely the total orientations of corner cubies is divisible by 3 and the total orientations of edge cubies is divisible by 2. Because the basic moves are the generators of $\mathbb{G}$, hence condition 3 holdes under the whole group action of $\mathbb{G}$.

Sufficient condition. Suppose a given configuration satisfies the 3 conditions, we can construct a sequence moves in $\mathbb{G}$ to transform it to the solved configuration, there are 4 steps

1. Recover the corner cubie positions: $\sigma^{-1}\sigma_0$ is an even permutation in A_{20}, which is generated by 3-cycles. Using conjugation and corner 3-cycle, Lem 3.3 and Lem 3.4, one can construct a sequence of moves to produce $\sigma^{-1}\sigma_0$. Hence, transform σ to σ_0.

2. Recover the edge cubie positions: $\tau^{-1}\tau_0$ is an even permutation in A_{30}, which is generated by 3-cycles. Using conjugation and edge 3-cycle, Lem 3.3 and Lem 3.5, one can construct a sequence of moves to produce $\tau^{-1}\tau_0$. Hence, transform τ to τ_0.
3. Recover the corner cubie orientations: choose the smallest two corner cubies, whose orientations are both non-zeros, (x_i, x_j), using conjugate and corner twisting operation in Lem 3.3 and Lem 3.6, to change x_i to be 0. Then the number of non-zero orientation corner cubies is reduced. Repeat this procedure, until all x_i's are zeros.
4. Recover the orientation of cubie orientations: choose the smallest two edge cubies, whose orientations are both non-zeros, (y_i, y_j), using conjugate and edge flipping operation in Lem 3.3 and Lem 3.9, to change y_i to be 0. Then the number of non-zero orientation edge cubies is reduced. Repeat this procedure, until all y_i's are zeros.

Then we reach the solved configuration. □

Corollary. *Given a solvable configuration, there is a deterministic algorithm to construct a sequence of moves to transform it to the initial configuration.*

Proof. The proof for the sufficient condition in theorem 3.8 gives the algorithm. □

3.5. Megaminx Group Structure

Let G_O denote the subgroup, each move in G_O fix the positions of all corner and edge cubies, but twist the orientations; G_P denote the subgroup, each move in G_P fix the orientations of all corner and edge cubies, but permutes their positions.

Lemma 3.9. *Let G_O denote the set of moves that fix the positions of all corner and edge cubies, then G_O is a normal subgroup of the megaminx group $\mathbb{G}$, which is called the orientation twisting subgroup.*

Proof. Construct a homomorphism $\varphi : \mathbb{G} \to S_{50}$, which maps a move in $\mathbb{G}$ to a permutation of all unoriented corner and edge cubies. Then it is obvious that $G_O = ker\ \varphi$, hence G_O is a subgroup.

Furthermore G_O is normal in $\mathbb{G}$. Suppose $n \in G_O$ and $g \in \mathbb{G}$, then $\varphi(n) = e \in S_{50}$. Consider the conjugate $g^{-1}ng$,

$$\varphi(g^{-1}ng) = \varphi(g)^{-1}\varphi(n)\varphi(g) = \varphi(g)^{-1}e\varphi(g) = e \in S_{50},$$

therefore $g^{-1}ng \in ker\ \varphi$, namely $g^{-1}ng \in G_O$, G_O is normal. □

Lemma 3.10. *The orientation twisting subgroup*

$$G_O = \mathbb{Z}_3^{19} \times \mathbb{Z}_2^{29},$$

where $\mathbb{Z}_m^n$ is the direct product of n copies of $\mathbb{Z}_m$.

Proof. We consider the orientations of 20 corner cubies. Suppose, there is a corner cubie C_i with non-zero orientation, because the total sum of all corner cubie orientations is 0, there must be another cornier cubie C_j with non-zero orientation. We apply the corner twist operation in Lem 3.6 and conjugation multiple times, we can change the orientation of C_i to be 0. Therefore, the total number of corner cubies with non-zero orientation is reduced. By repeating this procedure, we can change the orientations of all corner cubies to be zeros. Because the total orientation is 0, the orientation of the last corner cubie is determined by the first 19 cubie corners. This shows the corner orientation twisting subgroup is isomorphic to $\mathbb{Z}_3^{19}$.

Similarly, the edge orientation flipping subgroup is isomorphic to $\mathbb{Z}_2^{29}$. Because the whole group G_O is Abelian, both subgroups are normal, and G_O is the direct product of the two subgroups. □

Definition 3.3 (Normal Closure)**.** *The normal closure of a subset A of a group G, $N_{CL}(A)$ is the intersection of all normal subgroups in G that contain A,*

$$N_{CL}(A) = \bigcap_{A\subseteq N, N\trianglelefteq G} N.$$

Corollary. *The orientation twisting normal subgroup G_O is the normal closure of the set of corner twisting move Eqn. 2 and edge flipping move Eqn. 3*

$$\{[\sigma_1,\tau_1],[\sigma_2,\tau_2]\}$$

where

$$\sigma_1 = DF^2EF^{-1}D^{-1}, \tau_1 = A, \sigma_2 = BD^{-1}C^2E^{-1}F^{-1}C^2DB^{-1}, \tau_2 = A.$$

Proof. Combine Lem 3.6, Lem 3.9 and Lem 3.10. □

Lemma 3.11. *The position permutation subgroup G_P has the direct product structure*

$$G_P = A_{20} \times A_{30}.$$

Proof. The permuations of all the unoriented corner cubies are even, represented as A_{20}; the permuations of all the unoriented edge cubies are even, represented as A_{30}.

Suppose n permutes the corner cubie positions without changing the orientations, $n \in A_{20}$, and $g \in G_P$, then the support of n only contains corner cubies, the intersection of the support of n and that of g only contains corner cubies. Namely, no edge cubie is in the intersection, hence all the edge cubies will be fixed by $g^{-1}ng$. Therefore, $g^{-1}ng$ only permutes the corner cubies without changing the orienations, $g^{-1}ng \in A_{20}$. This means A_{20} is normal in G_P. Similarly A_{30} is normal, $A_{20} \cap A_{30} = e$. Furthermore, any permutation $g \in G_P$ can be decomposed into cycles, each cycle only contains corner cubies or edge cubies, hence belongs either to A_{20} or A_{30}. Hence $g = \alpha\beta$, where $\alpha \in A_{20}$ and $\beta \in A_{30}$. By definition of direct product, $G_P = A_{20} \times A_{30}$. □

Theorem 3.12 (Group Structure)**.** *Suppose $(\mathbb{G}, *)$ is the megaminx group, then it has the decomposition as the semidirect product:*

$$\mathbb{G} = G_O \rtimes G_P = (\mathbb{Z}_3^{19} \times \mathbb{Z}_2^{29}) \rtimes (A_{20} \times A_{30}). \tag{4}$$

Proof. By previous argument, Lem 3.10 shows G_O is a normal subgroup; the intersection between G_O and G_P is the identity of $\mathbb{G}$. Each move $M \in \mathbb{G}$ can be decomposed into the product of an orientation twising move and a position permutation move, namely $\mathbb{G} = G_O G_P$. Hence the megaminx group is the semidirect product of G_O and G_P. $\square$

Corollary. *The size of the megaminx group or the total number of solvable configurations is*

$$|\mathbb{G}| = \frac{1}{4} 3^{19} \times 2^{29} \times 20! \times 30!$$

Proof. By direct computation, using the group structure Eqn. 4 and the facts $|\mathbb{Z}_3^{19}| = 3^{19}$, $|\mathbb{Z}_2^{29}| = 2^{29}$, $|A_{20}| = 20!/2$ and $|A_{30}| = 30!/2$. $\square$

4. Conclusion

This work proves the structure of the megaminx group, gives the sufficient and necessary conditions for solvable configurations, and introduce a constructive algorithm to solve megaminx. In the future, the God's number of megaminx will be explored.

Acknowledgement

The author finds the topic, and the mentor suggests to use group theory. The mentor teaches the author the basic group theory. All the key operations, such as corner cubie 3-cycles, edge cubie 3-cycles, corner cubie twists and edge cubie flips and so on are figured out solely by the author.

References

[1] D. Joyner, Adventures in Group Theory: Rubik's Cube, Merlin's Machine, and Other Mathematical Toys, Johns Hopkins University Press, 2008.

[2] J. Chen, Group theory and the rubik's cube, Lecture Notes.

[3] T. Davis, Group theory via rubik's cube, Lecture Notes.

[4] J. G. Nourse, The Simple Solution to Rubik's Cube, Bantam, 1981.

[5] F. H. A. Jr., D. Singmaster, Handbook of Cubik Math, Enslow Publishers, 1982.

[6] D. Kunkle, G. Cooperman, Twenty-six moves suffice for rubiks cube, in: Proceedings of the 2007 international symposium on Symbolic and algebraic computation ISSAC07, 2007.

[7] T. Rokicki, H. Kociemba, M. Davidson, J. Dethridge, The diameter of the rubik's cube group is twenty, SIAM Journal of Discrete Mathematics 27 (2) (2010) 1082–1105.