参赛队员姓名： Cheng Qin

中学： 上海中学国际部

省份： 上海

国家/地区： 上海

指导教师姓名： 宋雷

指导教师单位： 中山大学数学学院

论文题目： Fourth Moments and Larsen’s Alternatives

Fourth Moments and Larsen's Alternative

Cheng Qin

Abstract

The representation theory of compact infinite groups is a well-known subject. However, attempts at classifying compact groups are explored to a lesser extent. In this article, we will be examining one such method of classification: Larsen's alternative. This will in turn alert us to the significance of fourth moments in determining the identity of compact groups. This article will define fourth moments and present the proof of Larsen alternative, following [4, 6]. We will also calculate the fourth moments of several important compact groups.

Keywords: Fourth Moments, Larsen's Alternative, Representation Theory

Contents

1 Introduction

Let $G \subset GL_n(\mathbb{C})$ be a compact matrix group, endowed with its Haar measure μ. We define the fourth moment of this inclusion as

$$M_4(G, GL_n(\mathbb{C})) = \int_G |\mathrm{Tr}(g)|^4 d\mu(g). \tag{1}$$

Using representation theory, we can give another expression of the fourth moment. In fact, let $V = \mathbb{C}^n$ be the natural representation of G. Then the fourth moment can be calculated by

$$M_4(G, GL_n(\mathbb{C})) = \dim_{\mathbb{C}} \mathrm{End}_G(\mathrm{End}(V)), \tag{2}$$

where $\mathrm{End}(V)$ is a representation of G induced by the representation of G on V, and $\mathrm{End}_G(\mathrm{End}(V))$ is the space of automorphisms of $\mathrm{End}(V)$ as G-representations. This result, which is proved in Proposition 3.1, tells us that the fourth moment is in fact a positive integer. In [4], Katz wrote down a prove of criterion, called Larsen's alternative, which, by calculating the fourth moments, determines the nature of the group. In this article, we shall follow the formulation and proof given in [6], to state and prove the Larsen's alternative.

Theorem (Larsen's Alternative) *Let $n \geqslant 2$. Let G be a compact subgroup of $SU(n)$. If the fourth moment of G is equal to 2 then either $G = SU(n)$ or G is a finite group.*

The proof of this theorem, following [6], is given in Section 3.2. There are applications of Larsen's alternative. See, for example, [4], for the application of Larsen's alternative on the study of Lefschetz pencils, and [6] for the application on Kloosterman sums.

The Larsen's alternative stresses the importance of the fourth moments. In this article, we calculate the fourth moment of some classical groups. The results we obtained are as follows.

Theorem *The permutation group S_n can be viewed as a subgroup of $GL_n(\mathbb{C})$ by realising the elements as permutation matrices. The fourth moment of this inclusion is $M_4(S_n, GL_n(\mathbb{C}) = 15$.*

Theorem *The fourth moments of finite subgroups of $\mathrm{SO}(3,\mathbb{R})$, viewed as subgroups of $GL_3(\mathbb{C})$, is as follows.*
(1) The fourth moment of the cyclic group with n elements C_n:

$$M_4(C_n, GL_n(\mathbb{C})) = \begin{cases} 41, & \text{if } n = 2 \\ 27, & \text{if } n = 3 \\ 21, & \text{if } n = 4 \\ 19, & \text{if } n > 4 \end{cases}.$$

(2) The fourth moment of the dihedral group with $n = 2k$ elements D_{2k}:

$$M_4(D_{2k}, GL_n(\mathbb{C})) = \begin{cases} 21, & \textit{if } n = 2 \\ 14, & \textit{if } n = 3 \\ 11, & \textit{if } n = 4 \\ 10, & \textit{if } n > 4 \end{cases}.$$

(3) The fourth moment of the tetrahedral group $T \cong A_4$ *is* 7.
(4) The fourth moment of the octahedral group $O \cong S_4$ *is* 4.
(5) The fourth moment of the icosahedral group $I \cong A_5$ *is* 3.

The detail of the calculation is exhibited in Section 4.4.

We also calculate the fourth moment of $\mathrm{O}(n, \mathbb{R}) \subset GL_n(\mathbb{C})$ and $\mathrm{SO}(n, \mathbb{R}) \subset GL_n(\mathbb{C})$.

Theorem *Let* $n \geq 2$ *be an positive integer.*
(1) The fourth moment of $\mathrm{O}(n, \mathbb{R}) \subset GL_n(\mathbb{C})$ *is* 3 *for any* n.
(2) $M_4(\mathrm{SO}(n, \mathbb{C}), GL_n(\mathbb{C})) = \begin{cases} 6, & \textit{if } n = 2 \\ 4, & \textit{if } n = 4 \\ 3, & \textit{if } n \neq 2, 4 \end{cases}.$

This result is obtained by examining carefully the irreducible decomposition of $\mathrm{End}(V)$ as $O(n, \mathbb{R})$-represenetaions (resp. $SO(n, \mathbb{R})$-representations). See Section 4.5 for detail. By this result, we can get the following result whose statement does not require representation theory and which is of its own interest.

Corollary 1.1 *Let*

$$O(n)^- := \{g \in O(n, \mathbb{R}) : \det g = -1\}$$

be endowed with the measure induced from the Haar measure of $O(n, \mathbb{R})$. *Then*

$$\int_{O(n)^-} |\mathrm{Tr}(g)|^4 dg = \begin{cases} 0, & \textit{if } n = 2 \\ 1, & \textit{if } n = 4 \\ 3/2, & \textit{if } n \neq 2, 4 \end{cases}. \tag{3}$$

The structure of this article is as follows. In Section 2, we will recall some basic results of representation theory of finite groups, compact matrix groups and Lie algebras. The content is totally classical and we refer to [1, 2, 3, 5] for main references. In Section 3, we shall present the fourth moment and the proof of Larsen's alternative following [4, 6]. In the last section, we will calculate in detail the fourth moment of some classical groups, using principally representation theory.

2 Representation Theory

In this section, we will present some basic knowledge in representation theory which will served as a tool to the proof of Larsen's alternative (Section 3) and the calculation of the fourth moments (Section 4). We will in particular present the representation theory of finite groups and character theory [3], the representation theory of compact groups [5] and of Lie algebras [2].

2.1 Group Representations

Definition 2.1 Let G be a group. A k-representation of G is the data of a k-vector space V and of a group homomorphism $\rho : G \to GL(V)$.

Remark 2.1 For the sake of convenience, one often writes $g.v$ for $\rho(g)v$. Then the definition of representation says that $(gh).v = g.(h.v)$.

Definition 2.2 Let V be a representation of a group G, W is called a subrepresentation if for all $g \in G$ and $w \in W$, $gw \in W$

Definition 2.3 A representation V of a group G is called irreducible if the only subrepresentations of V are $\{0\}$ and itself.

Definition 2.4 Let G be a group. Let V and W both be representations of G. A map $\phi : V \to W$ is called a homomorphism of G-representations, if:
1. $\phi : V \to W$ is a linear map, and
2. For all $g \in G$, $v \in V$, $\phi(gv) = g\phi(v)$.

Remark 2.2 Let $\phi : V \to W$ be a homomorphism of $G - representations$. Then $\ker\phi$ and $\mathrm{Im}\phi$ are subrepresentations of V and W respectively.

Definition 2.5 Two G-representations V and W are isomorphic to one another if there exists an homomorphism between them $\phi : V \to W$ that is invertible. This will be denoted as $V \cong W$.

2.2 Schur's Lemma

Lemma 2.1 (Schur) *Let G be a group and V and W be irreducible, finitely dimensional representations of G. Let $\phi : V \to W$ be a homomorphism between representations. Then:*
1. *Either ϕ is an isomorphism or $\phi = 0$.*
2. *Assume k is an algebraically closed field (e.g. $k = \mathbb{C}$) If $V = W$ acting on k, then $\phi = \lambda Id_V$.*

Proof Assume that $\phi \neq 0$, then $\ker\phi \neq V$. By remark 2.2, $\ker\phi$ is a subrepresentation of V. Since V is irreducible, and $\ker\phi \neq V$, then $\ker\phi = \{0\}$. So, ϕ is injective. We also note that $\mathrm{Im}\phi \neq 0$ and W is irreducible. By remark 2.2, since $\mathrm{Im}\phi$ is a subrepresentation of W, $\mathrm{Im}\phi = W$. So ϕ is surjective. Since ϕ is both injective and surjective, it is bijective and thus is an isomorphism.

To prove 2, let $\phi : V \to V$ be a linear map. Then we know that ϕ is bijective if and only if $\det\phi \neq 0$. Let us now examine the $\phi - \lambda Id$: since k is algebraically closed and the determinant function is essentially a polynomial for λ, there exists $\lambda \in k$ such that $\det(\phi - \lambda Id) = 0$. For this value of $\lambda \in k$, $\phi - \lambda Id$ is thus not an isomorphism.

Since it is clear that $\phi - \lambda Id : V \to V$ is a homomorphism of representations, but $\phi - \lambda Id$ is not an isomorphism, so $\phi - \lambda Id = 0$ i.e. $\phi = \lambda Id$.

□

Corollary 2.1 *Let G be an Abelian group, let V be a k-representation of G. If V is irreducible, then* $\dim(V) = 1$.

Proof Let $g \in G$. We will show that $\rho(g) : V \to V$ is a homomorphism of representations, where $\rho(g)$ is the representation matrix of g, in fact, by the commutativity of G:

$$\forall h \in G, v \in V, \rho(g)(hv) = h\rho(g)v = h\rho(g)(v).$$

So $\rho(g)$ is a homomorphism of representations. Since V is irreducible, by Schur's lemma we have $\rho(g) = \lambda Id$ for some constant λ. Hence, all values $h \in G$ act on V as scalars.

Hence if $v \neq 0, v \in V$, then hv $(\forall h \in G)$ is a 1 dimensional subspace spanned by a single line described by V. However, we know that V is irreducible, hence $\dim(V) = 1$.

□

2.3 Maschke's Theorem

Let V be a $\mathbb{C}$-representation of a finite group G.

Definition 2.6 Let $(\cdot, \cdot)$ be a Hermitian form on V. This form is called G-stable if for all $g \in G$ and $x, y \in V$, we have $(gx, gy) = (x, y)$.

Lemma 2.2 *Let $(\cdot, \cdot)$ be a G-stable Hermitian form on V. Let $W \subset V$ be a subrepresentation, then $W^{\perp}$ is a subrepresentation of V.*

Proof Let $w \in W$ and $w' \in W^{\perp}$, we want to prove that for all $g \in G$, $gw' \in W'$. Since for all $w \in W$, $gw \in W$, there exists g such that $w = g \cdot w_i$, $w_i \in W$.

$$(w, g \cdot w') = (g \cdot w_i, g \cdot w') = (w_i, w') = 0$$

Hence, for all w', $gw' \in W'$ and hence $W^{\perp}$ is a subrepresentation of V.

□

Lemma 2.3 *Let V be a non-irreducible representation of finite group G so that ther exists a subrepresentation $W \subset V$ where $W \neq 0, V$. Then there exists $W' \subset V$ such that $V = W \oplus W'$.*

Proof Let $(\cdot, \cdot)$ be a Hermitian form (not necessarily G-stable) on V. Let us define another form $(\cdot, \cdot)'$ as for all $x, y \in V$,

$$(x, y)' = \frac{1}{|G|} \sum_{g \in G} (gx, gy)$$

.

We can verify that $(\cdot, \cdot)'$ is indeed a Hermitian form.

We will now try to prove $(\cdot, \cdot)'$ is G-stable:

$$(gx, gy)' = \frac{1}{|G|} \sum_{h \in G} (hgx, hgy) = \frac{1}{|G|} \sum_{k \in G} (kx, ky) = (x, y)'$$

. Hence $(x, y)'$ is G-stable.

Since W is a subrepresentation of V and $(\cdot, \cdot)'$ is G-invariant, we may apply Lemma 2.2 to show that there exists a subrepresentation $W' = W^{\perp} \subset V$ satisfying $V = W \oplus W^{\perp}$. Hence, lemma 2.3 is proved. □

Theorem 1 (Maschke) *Let V be a representation of finite group G. Then V can be written as a direct sum of irreducible representations:*

$$V = \bigoplus_{i=1}^{r} \lambda_i V_i$$

Where $\lambda_i,\ r \in \mathbb{N}$. Furthermore, this decomposition is unique up to ordering and isomorphism.

Proof Suppose V is irreducible. Then $V = V$, which is a unique direct sum of irreducible representations.

If V is not irreducible, by Lemma 2.3, we can always find non-trivial subrepresentations w and W' such that $V = W \oplus W'$. We will now prove the existence of the decomposition by induction on the dimension of the representation:

Base Case: $\dim(V) = 1$, then V is irreducible and $V = V$ is the irreducible decomposition:

Assume $\dim(V) = n$. Assume that any representation of dimension less than n has an irreducible decomposition. Then, by the claim, $V = W \oplus W'$ where $\dim(W)$ and $\dim(W')$ are less than n.

By induction, W and W' will decompose into irreducible representations, which in turn gives the irreducible decomposition of V. Hence, we have proved the existence of the irreducible decomposition of representations V of finite group G.

Now, we will prove the uniqueness of this decomposition. Let

$$V = \bigoplus_{i=1}^{r} \lambda_i V_i = \bigoplus_{i=1}^{s} \mu_i W_i.$$

be two irreducible representations. We want to show that for all V_i, there exists W_j such that $V_i \cong W_j$. Define:

$$\rho_j : V_i \to V \to W_j.$$

One checks easily that ρ_j is a homomorphism of representations. We note here that V_i and W_j are irreducible. Then, by Schur's lemma, either $\rho_j = 0$ or ρ_j is an isomorphism. In the latter case we are done.

We discount the case where all $\rho_j = 0$ since then the first part of the map must also be 0, which is impossible since $V_j \neq \{0\}$.

□

2.4 Character Theory

Definition 2.7 Let (V, ρ) be a $\mathbb{C}-$representation of G. The character of (V, ρ) is a map

$$\chi_V : G \to C$$
$$g \mapsto \mathrm{Tr}(\rho(g)).$$

Lemma 2.4 *Let V, W be representations of G, then $V \oplus W$ is also a representation. Then $\chi_{V \oplus W} = \chi_V + \chi_W$*

Proof Let $dim(V) = n$ and $dim(W) = m$, then $dim(V \oplus W) = m+n$. Here we note that V and W have two disjoint sets of basis, meaning that $\rho_v(g)$ cannot affect any of the basis of W and vice versa. Then it is obvious that $\rho_{V \oplus W}(g)$ consists of a direct sum of matrices $\rho_V(g)$ and $\rho_W(g)$, hence proving the lemma. □

Lemma 2.5 *If $g, g' \in G$ are conjugates to each other (i.e. $\exists h \in G$ s.t. $g' = hgh^{-1}$), then $\chi_V(g) = \chi_V(g')$*

Proof

$$\begin{aligned}\chi_V(g') = \mathrm{Tr}(\rho(g')) = \mathrm{Tr}(hgh^{-1}) = \mathrm{Tr}(\rho(h)\rho(g)\rho(h^{-1})) \\ = \mathrm{Tr}(\rho(h)\rho(h^{-1})\rho(g)) = \mathrm{Tr}(\rho(g)) = \chi_V(g).\end{aligned}$$

□

Definition 2.8 Let V be a representation of G, we define

$$V^G := \{v \in V : gv = v, \ \forall g \in G\}.$$

We note here that V^G is a subrepresentation of V.

Proposition 2.1 *Let V and W be G-representations. One can equip $\mathrm{Hom}(V, W)$ with a G-representation structure by defining a map:*

$$g.\phi : \ V \to W$$
$$g.\phi : \ v \mapsto g\phi(g^{-1}v).$$

for $g \in G$ and $\phi \in \mathrm{Hom}(V, W)$.

Lemma 2.6 $\mathrm{Hom}(V, W)^G$ *consists of linear maps that are homomorphisms of representations: $V \to W$.*

Proof Let $h, g \in G$, and map $g.\rho : \ V \to W$ be defined as above. Then:

$$(hg).\rho = (hg) \cdot \rho((hg)^{-1}) = (hg) \cdot \rho(g^{-1}h^{-1} = h \cdot ((g\rho(g^{-1})h^{-1}) = h.(g.\rho).$$

We will now verify that under the above definition for $g.\rho$, $Hom(V, W)^G$ consists of homomorphisms: $V \to W$:

$$\rho(gv) = g \cdot \rho(g^{-1}(gv)) = g \cdot \rho(v).$$

□

Lemma 2.7 *Let V be a representation of a finite group G and χ_V be the corresponding character. Then:*

$$\frac{1}{|G|}\sum_{g\in G}\chi_V = \dim(V^G).$$

Proof

$$\frac{1}{|G|}\sum_{g\in G}\chi_V = \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(\rho(g)) = \mathrm{Tr}\left(\frac{1}{|G|}\sum_{g\in G}\rho(g)\right).$$

Let

$$A = \frac{1}{|G|}\sum_{g\in G}\rho(g).$$

Let $v \in V^G$, then

$$Av = \frac{1}{|G|}\sum_{g\in G}\rho(g)v = \frac{1}{|G|}\sum_{g\in G}v = v.$$

Then for all $v \in V,\ Av \in V^G, g \in G$, we have:

$$gAv = g\cdot\frac{1}{|G|}\sum_{h\in G}\rho(h)v = \cdot\frac{1}{|G|}\sum_{h\in G}\rho(gh)v = \frac{1}{|G|}\sum_{k\in G}\rho(k)v = Av.$$

Therefore, A is of the form $\left(\begin{array}{c|c} 0 & * \\ \hline 0 & Id_{dim(V^G)}\end{array}\right)$, hence $\mathrm{Tr}A = \dim(V^G)$.

□

Lemma 2.8 *Let V, W be representations of G, then:*

$$\chi_V(g)\overline{\chi_W(g)} = \chi_{\mathrm{Hom}(V,W)}(g).$$

Proof Let $g \in G$ and $\rho_V(g)$ denote the matrix associated with g acting on representation V and correspondingly for $\rho_W(g)$. We note that $\overline{\chi_W(g)} = Tr(\overline{\rho_W(g)})$. We also know that for some k, $\rho_W(g)^k = Id$ since G is a finite group. Hence, we know by the Jordan normal form, that that $\rho_W(g)$ is conjugate to a diagonal matrix $diag(\lambda_1, ..., \lambda_n)$ with $|\lambda_i| = 1$. Let $\dim(V) = n$ and $\dim(W) = m$. Then the trace of $\rho_W(g)$ can be written as:

$$Tr(\rho_W(g)) = \sum_{i=1}^{n}\lambda_i,\ |\lambda_i| = 1.$$

Then since $|\lambda_i| = 1$, $\mathrm{Tr}(\overline{\rho_W(g)}) = \mathrm{Tr}((\rho_W(g))^{-1}) = \mathrm{Tr}(\rho_W(g^{-1}))$.

Now, choose bases for $W:\ \{e_1,\ ..., e_m\}$ and for $V:\ \{f_1,\ ..., f_n\}$ such that $\rho_W(g)e_i = \lambda_i e_i$ and $\rho_V(g)f_j = \mu_j f_j$. Then we have a basis of $\mathrm{Hom}(V,W)$: $\{\rho_{ij}\}$ defined to be:

$$\rho_{ij}(e_i) = f_j\ ,\rho_{ij}(e_k) = 0\ \forall k \neq i.$$

Then, by Lemma 2.6, we have:

$$(\rho_{\mathrm{Hom}(V,W)}(g)\rho_{ij})(e_k) = \rho_V(g)\rho_{ij}(\rho_W(g^{-1})(e_k))$$
$$= \rho_V(g)(\rho_{ij}(\lambda_k^{-1}(e_k))) = \begin{cases} \lambda_i^{-1}\mu_j f_j \text{ if } k = i \\ 0 \text{ if } k \neq i \end{cases}.$$

Then, when $k = i$, we have:

$$\rho_{Hom(V,W)}(g)\rho_{ij} = \lambda_i^{-1}\mu_j f_j.$$

When $k \neq i$, we simply have 0.
Therefore, $\rho_{Hom(V,W)}(g)$ is conjugate to a Jordan normal form matrix whose diagonal consists of $\lambda_i^{-1}\mu_j$. Then:

$$\chi_{Hom(W,V)} = Tr(\rho_{Hom(W,V)}(g)) = \sum_{i,j} \lambda_i^{-1}\mu_j = \sum_{i,j} \overline{\lambda_i}\mu_j = \sum_{i=1}^{n} \lambda_i \cdot \sum_{j=1}^{m} \mu = \overline{\chi_W(g)}\chi_V(g).$$

□

Definition 2.9 Let V, W be representations of finite group G. Let χ_V, χ_W be their corresponding characters. Define the inner product of χ_V, χ_W as:

$$\langle \chi_V, \chi_W \rangle := \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\overline{\chi_W(g)}$$

Theorem 2 (First Orthogonality Relation) *Let V, W be irreducible representations of G, then:*

$$\langle \chi_V, \chi_W \rangle = 1, \text{ if } V \cong W,$$
$$\langle \chi_V, \chi_W \rangle = 0, \text{ if } V \ncong W.$$

Proof By definition, we have:

$$\langle \chi_V, \chi_W \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\overline{\chi_W(g)}.$$

By Lemma 2.8, we have:

$$\frac{1}{|G|} \sum_{g \in G} \chi_V(g)\overline{\chi_W(g)} = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{Hom}(V,W)}(g).$$

By Lemma 2.7, we have:

$$\frac{1}{|G|} \sum_{g \in G} \chi_{Hom(V,W)^G}(g) = \dim(\mathrm{Hom}(V,W)^G).$$

Then, putting it all together and applying Schur's Lemma, we have:

$$\langle \chi_v, \chi_w \rangle = \dim(Hom(V,W)^G) = 1 \text{ if } W \cong V,$$
$$\langle \chi_v, \chi_w \rangle = \dim(Hom(V,W)^G) = 0 \text{ if } W \ncong V.$$

□

Corollary 2.2 *Let V be a representation of G. Let $V =_1 V_1 \oplus ... \oplus_r V_r$ where m_i refers to the number of copies of irreducible representation V_i in the irreducible decomposition of V. Then:*

$$\langle \chi_V, \chi_V \rangle = \sum_{i=1}^{r} m_i^2.$$

Proof By Lemma 2.5 and the First Orthogonality Relation, we have:

$$\langle \chi_V, \chi_V \rangle = \sum_{i,j} \langle m_i \chi_{V_i}, m_j \chi_{V_j} \rangle = \sum_{i=1}^{r} m_i^2.$$

□

Corollary 2.3 *Let V be a representation of G. Then V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.*

Proof

$$\langle \chi_V, \chi_V \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_V(g)} = \frac{1}{|G|} \sum_{g \in G} \chi_{End(V)}(g) = dim(End(V)^G).$$

By Lemma 2.7, we know that $End(V)^G$ consists of the linear map that are are homomorphisms of representations: $V \to W$. Since V is irreducible, by Schur's Lemma, we have $End(V)^G = \mathbb{C}Id$, and hence, $\dim(End(V)^G) = 1$.

Suppose $\langle \chi_V, \chi_V \rangle = 1$. Then by Corollary 2.2, we have $\langle \chi_V, \chi_V \rangle = a_1^2 + ... + a_r^2$ where $V = a_1 V_1 + ... + a_r V_r$. However, since $\langle \chi_V, \chi_V \rangle = 1$, that would mean that the irreducible decomposition of V can only consist of one irreducible representation. Hence, V is irreducible. □

2.5 Lie Algebras

Definition 2.10 A Lie Algebra is a vector space L over a k endowed with a bilinear form $[\cdot, \cdot] : L \times L \to L$ satisfying:

1. For all $x, y \in L$, $[x, y] = -[x, y]$
2. For all $x, y, z \in L$, $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ (Jacobi Identity)

Example 2.1 Let $L = M_n(k)$ be the vector space of all $n \times n$ matrices with entries in k. Let $[A, B] := AB - BA$. Then, $(L, [\cdot, \cdot])$ is a Lie algebra.

Definition 2.11 Let $(L_1, [\cdot, \cdot]_1)$ and $(L_2, [\cdot, \cdot]_2)$ be two Lie algebras. A homomorphism of Lie algebras is a linear map $\phi : L_1 \to L_2$ satisfying for all $x, y \in L_1$ $[\phi(x), \phi(y)]_2 = \phi([x, y]_1)$.

Definition 2.12 A subalgebra of a Lie algebra $(L, [\cdot, \cdot])$ is a subspace L' such that for all $x, y \in L'$, $[x, y] \in L'$.

Lemma 2.9 *The image of a homomorphism of lie algebras $\phi : L_1 \to L_2$ is a subalgebra.*

Proof Let $x, y \in L_1$ and $\phi(x), \phi(y) \in L_2$. Then since ϕ is a homomorphism of lie algebras: $\phi([x, y]) = [\phi(x), \phi(y)] \in L_2$.

$\square$

Definition 2.13 An ideal of a Lie algebra $(L, [\cdot, \cdot])$ is a subspace I such that for all $x \in L, y \in I,\ [x, y] \in I$.

Lemma 2.10 *The kernel of a homomorphism of Lie algebras $\phi : L_1 \to L_2$ is an ideal of L_1.*

Proof Let $x \in L_1$, $y \in \ker(\phi)$, and $\phi(x), \phi(y) \in L_2$. Then, $\phi([x, y]) = [\phi(x), \phi(y)] = [\phi(x), 0] = 0$.

$\square$

Definition 2.14 Let $(L, [\cdot, \cdot])$ be a Lie Algebra. A representation of L is the data of a vector space V and a homomorphism of Lie algebras $\rho : L \to \mathrm{End}(V)$.

Remark 2.3 Similar to finite group theory, we can define subrepresentations, homomorphisms and isomorphisms between representations, irreducibility, and even Schur's lemma for Lie algebras. However, Maschke's theorem doesn't exist in general for Lie algebras.

2.6 Compact Matrix Groups

Example 2.2 $SO_n(\mathbb{R})$ and $O_n(\mathbb{R})$ are compact.

Proof It suffices to prove that $O_n(\mathbb{R})$ is compact. Let $A \in O_n(\mathbb{R})$. Then $AA^\intercal = Id$ can be represented as:

$$\begin{pmatrix} a_{11} & a_{12} & \cdot & a_{1n} \\ \cdot & \cdot & & \cdot \\ a_{i1} & a_{i2} & \cdot & a_{in} \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \cdot & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & \cdot & a_{i1} & \cdot & a_{n1} \\ a_{12} & \cdot & a_{i2} & \cdot & a_{n2} \\ \cdot & & \cdot & & \cdot \\ a_{1n} & \cdot & a_{in} & \cdot & a_{nn} \end{pmatrix} = Id_n$$

Hence $\forall i \in \{1, 2, ..., n\}$, $a_{i1}^2 + a_{i2}^2 ... + a_{in}^2 = 1$ and hence the absolute value of any entry in any $A \in O_n(\mathbb{R})$ is bounded by 1.

$\square$

Example 2.3 $SU_n(\mathbb{C})$ and $U_n(\mathbb{C})$ are compact.

Proof It suffices to prove that $U_n(\mathbb{C})$ is compact. Let $A \in U_n(\mathbb{C})$. Then representing $A\overline{A^\intercal} = Id$ can in a similar way to Theorem 5, we have $\forall i \in \{1, 2, ..., n\}$:

$$\sum_{j=1}^{n} a_{ij}\overline{a_{ij}} = \sum_{j=1}^{n} |a_{ij}|^2 = 1 \Rightarrow |a_{ij}| \leqslant 1$$

$\square$

Definition 2.15 Let G be a group. A measure μ on G is called left invariant if for all $g \in G$, measurable subsets $A \subset G$, $\mu(gA) = \mu(A)$.

Remark 2.4 The definition for right invariant measures is similar.

Proposition 2.2 ([5]) *Any matrix group admits a left-invariant measure.*

Definition 2.16 A Haar measure is a measure of a group G that is both left and right invariant, with $\mu(G) = 1$.

Theorem 3 ([5]) *Let G be a compact matrix group. Then there exists a unique Haar measure on G.*

Remark 2.5 The representation theory of compact groups are quite similar with that of finite groups. This is reflected in the simple replacement of $\frac{1}{|G|}\sum_{g\in G}\dots$ by $\int_A \dots d\mu$, where μ represents the Haar measure of a compact group, in the proof of Schur's Lemma, Maschke's theorem, the First Orthogonality Relation, and all associated corollaries and lemmas during the proofs of these theorems which all apply to compact matrix groups.

Definition 2.17 Let G be a matrix group. We define the tangent space of G at the identity element e as:

$$T_eG := \{A \in M_n(\mathbb{C}) : \exists\ \varepsilon \in \mathbb{R}, \delta : (-\varepsilon, \varepsilon) \to G,\ s.t.\ \delta(0) = Id,\ \delta'(0) = A\}$$

Example 2.4 Let $G = SU(n)$, then $T_e(G) = \{A \in M_n(\mathbb{C}) : A + \overline{A^\intercal} = 0,\ A = 0\}$

Example 2.5 Let $G = SL_n(\mathbb{C})$, then $T_e(G) = \{A \in M_n(\mathbb{C}) : \text{Tr } A = 0\}$.

Proposition 2.3 *Let $G \subset GL_n(\mathbb{C})$ be a matrix group. Knowing that $T_eG \subset M_n()$ has a Lie algebra structure defined by $[A, B] := AB - BA$, then $T_eG \subset M_n(\mathbb{C})$ is a sub Lie algebra.*

Proof For $X, Y \in T_eG$, we need to show that $[X, Y] = XY - YX \in T_eG$. Let $X = \delta'(0)$ and $Y = \mu'(0)$, where δ and η are paths from $(-\varepsilon, \varepsilon) \to G$. For a fixed $s \in (-\varepsilon, \varepsilon)$ consider the path:

$$\Gamma_s : (-\varepsilon, \varepsilon) \to G$$
$$t \mapsto \delta(t)\eta(s)\delta^{-1}(t)$$

Then, by the product rule of derivatives, we have:

$$\Gamma'(0) = \delta'(0)\eta(s)\delta^{-1}(0) + \delta(0)\eta(s)\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\delta^{-1}(t) = X\eta(s) - \eta(s)\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\delta^{-1}(t).$$

The proof is finished by the following lemma:

Lemma 2.11

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \delta^{-1}(t) = -X.$$

Proof We note that $Id = \delta(t)\delta^{-1}(t)\ \forall t \in (-\varepsilon, \varepsilon)$. Then, differentiating both both sides with t, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t} Id = \delta'(t)\delta^{-1}(t) + \delta(t) \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \delta^{-1}(t).$$

Setting $t = 0$ and then substituting for desired variables:

$$0 = \delta'(0)\delta^{-1}(0) + \delta(0) \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \delta^{-1}(t) = X + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \delta^{-1}(t) \Rightarrow \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \delta^{-1}(t) = -X.$$

□

Hence, for all $s \in \varepsilon$, $X\eta(s) - \eta(s)X \in T_eG$. Therefore,

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} X\eta(s) - \eta(s)X = X\eta'(0) - \eta'(0)X = XY - YX \in T_eG.$$

□

Proposition 2.4 *Let V be a representation of G which induces the representation of $\mathfrak{g}$ as described above. If V is irreducible as a $\mathfrak{g}-$representation, then it is irreducible as a $G-$representation.*

Proof Assume V is not irreducible as a G-representation. Let $W \subset V$ be a non-trivial G-representation, then for all $g \in G,\ w \in W$, $g.w \in W$.

Claim 2.1 $W \subset V$ is also a non-trivial sub-representation of $\mathfrak{g}$

We need to show that $\forall X \in \mathfrak{g}$, $w \in W$, then $X.w \in W$. Let $X = \delta'(0)$ with $\delta : (-\varepsilon, \varepsilon) \to G$. By (*), $\forall t \in (-\varepsilon, \varepsilon)$, $\delta(t)w \in W$. Then, $X.w = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \delta(t)w \in W$. However, this claim contradicts the irreducibility of V as a $\mathfrak{g}$-representation and thus the proposition is proved.

□

3 Larsen's Alternative

3.1 Fourth Moments

Definition 3.1 Let $G \subset GL_n(\mathbb{C})$ be a compact matrix group. The fourth moment of $G \subset GL_n(\mathbb{C})$ is $M_4(G, GL_n(\mathbb{C})) = \int_G |Tr(g)|^4 d\mu(g)$ where μ is the Haar measure of G.

Proposition 3.1 *We can regard $V = \mathbb{C}^n$ as a representation of G. Then $M_4(G, GL(V)) = \dim[\mathrm{End}(\mathrm{End}(V))^G]$.*

Proof By definition of fourth moment, we have:

$$M_4(G, GL(V)) = \int_G |\chi_V(g)|^4 d\mu(g) = \int_G \left(\chi_V(g)\overline{\chi_V(g)}\right)^2 d\mu(g)$$

Using we can use Lemma 2.8 and 2.9 to write:

$$\int_G \left(\chi_V(g)\overline{\chi_V(g)}\right)^2 d\mu(g) = \int_G (\chi_{\mathrm{End}(V)}(g))^2 d\mu(g) = \int_G \chi_{\mathrm{End}(\mathrm{End}(V))}(g) d\mu(g)$$
$$= \dim(\mathrm{End}(\mathrm{End}(V))^G)$$

□

Remark 3.1 If $End(V) = \bigoplus_{i=1}^r m_i V_i$ where all V_i are irreducible and not isomorphic to one another, then by Corollary 2.2, we have $M_4(G, GL_n(\mathbb{C})) = \sum_{i=1}^r m_i^2$.

Proposition 3.2 *Let $H \subset G \subset GL_n(\mathbb{C})$. Then $M_4(G, GL_n(\mathbb{C})) \leqslant M_4(H, GL_n(\mathbb{C}))$.*

Proof Let $V = \mathbb{C}^n$ be the representation of G and $End(V_G) = \bigoplus_{i=1}^r m_i V_i$ where V_i are irreducible representations of G. Then, by the remark, $M_4(G, GL_n(\mathbb{C})) = \sum_{i=1}^r m_i^2$. As H-representations however, each V_i may not be irreducible and may be further decomposed into $V_i = \bigoplus_{j=1}^{t_i} n_{ij} V_{ij}$. Then:

$$\mathrm{End}(V_H) = \bigoplus_{i=1}^r \bigoplus_{j=1}^{t_i} m_i(n_{ij} V_{ij})$$

And therefore:

$$M_4(H, GL_n(\mathbb{C})) = \sum_{i=1}^r \sum_{j=1}^{t_i} (m_i n_{ij})^2 \geqslant \sum_{i=1}^r m_i^2 = M_4(G, GL_n(\mathbb{C}))$$

□

3.2 Larsen's Alternative

The main references for this part is [4, 6]. The formalisation and proof is following [6].

Theorem 4 (Larsen's Alternative [6]) *Let $n \geqslant 2$. Let G be a compact subgoup of $SU(n)$. If the fourth moment of G is equal to 2: then either $G = SU(n)$ or G is a finite group.*

Proof Let $V = \mathbb{C}^n$. As G-representations, $End(V) = \bigoplus_{i=1}^r m_i V_i$. By remark 3.1, $M_4(G, GL_n(\mathbb{C})) = \sum_{i=1}^r m_i^2$. Then, since $M_4(G, GL_n(\mathbb{C})) = 2$, $End(V) = V_1 \oplus V_2$, where $V_1 \ncong V_2$ and V_1, V_2 irreducible (*).

In fact, $End(V) = End^0(V) \oplus W$ where $End^0(V) := \{\phi : V \to V, Tr\ \phi = 0\}$ and $W := \{\phi = \lambda Id_V : \lambda \in \mathbb{C}\}$. Taking into account (*), we conclude that $End^0(V)$ and W are irreducible representations.

Lemma 3.1 *As $SL_n(\mathbb{C})$-representations, $End^0(V)$ is defined as: for all $g \in GL_n(\mathbb{C})$, $v \in End^0(V) : g.v = gvg^{-1}$*

Proof Let $g, h \in G$ and $v \in End^0(V)$. Then:

$$h.(g.v) = h.(gvg^{-1}) = hgvg^{-1}h^{-1} = (hg).v$$

$$Id_G.v = Id.v = v$$

□

Lemma 3.2 *The induced representation of $\mathfrak{g} = \mathfrak{sl_n}(\mathbb{C})$ is defined by for all $X \in \mathfrak{sl_n}(\mathbb{C})$, $v \in End^0(V) = \mathfrak{sl_n}(\mathbb{C})$: $X.v = [X, v]$.*

Proof Say $X = \delta'(0)$, where $\delta : (-\varepsilon, \varepsilon) \to SL_n(\mathbb{C})$. By lemma 3.1, we have $\delta(t).v = \delta(t)v\delta(t)^{-1}$. By the definition of the induced representation:

$$X.v = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \delta(t).v = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \delta(t)v\delta(t)^{-1} = Xv - vX = [X, v]$$

□

We know that:

$$End^0(V) = \mathfrak{sl}_n(\mathbb{C}) = \mathfrak{su}(n) \otimes \mathbb{C}$$

We know that since $G \subset SU(n) \subset SL_n(\mathbb{C})$, $T_eG \subset \mathfrak{su}(n) \subset \mathfrak{sl}_n(\mathbb{C})$, then $T_eG \otimes_{\mathbb{R}} \mathbb{C} \subset \mathfrak{su}(n) \otimes_{\mathbb{R}} \mathbb{C}$, the latter of which is equal to $End^0(V)$.

Since the action of G on $T_eG \otimes_{\mathbb{R}} \mathbb{C}$ is a subrepresentation of the action of G on $\mathfrak{su}(n) \otimes_{\mathbb{R}} \mathbb{C} = End^0(V)$

By the irreducibility of $End^0(V)$ as a G-module, either $T_eG = \{0\}$ (1) or $End^0(V) = \mathfrak{su}(n) \otimes_{\mathbb{R}} \mathbb{C} = T_eG \otimes_{\mathbb{R}} \mathbb{C} \Rightarrow T_eG = \mathfrak{su}(n)$ (2).

(1) If $T_eG = \{0\}$, then G is discrete. But since G is compact, we conclude that G must be finite.

(2) If $T_eG = \mathfrak{su}(n)$, then since $G \subset SU(n)$, we have $G = SU(n)$.

□

4 Calculation of Fourth Moments

The Larsen's alternative places the importance of the fourth moments. In this section, we will calculate the fourth moments of some classical groups. Although the definition of the fourth moments is stated without any representation theory, its calculation, apart from some easy cases (See Section 4.1 and 4.3), demands representation theory. Our calculation of the fourth moments using representation theory also applies to the integration over some symmetric spaces (Corollary 4.1). To explore if our results can be applied in a similar way to other integration results would be an interesting topic.

4.1 Fourth moment of $SO_2(\mathbb{R})$ acting on $\mathbb{R}^2$

Let us first prove a lemma:

Lemma 4.1 *There exists a group isomorphism between* $S^1 := \{z \in \mathbb{C}^x : |z| = 1\}$ *and* $SO_2(\mathbb{R})$.

Proof It is easy to check both S^1 and $SO_2(\mathbb{R})$ are both groups. Now, using the fact that $\forall M \in SO_2(\mathbb{R})$, $det\ M = 1$ and $M^\intercal M = Id$ we will attempt to standardize the form of M. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \subset SO_2(\mathbb{R})$. Then using the two facts above:

$$ad - bc = 1,\ a^2 + b^2 = 1,\ c^2 + d^2 = 1,\ ac + bd = 0.$$

From these 4 equations, one can quickly realize that $a = d$ and $b = -c$, which in turn creates the motivation to write M as $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$, for some $\theta \in [0, 2\pi)$.

It is well known that elements in S^1 can be written in the form $\cos(\theta) + i\sin(\theta)$. Now, let us construct a map:

$$\rho : S^1 \to SO_2(\mathbb{R})$$

$$\cos(\theta) + i\sin(\theta) \mapsto \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta & \cos(\theta) \end{pmatrix}$$

Let $z_1 = \cos(\theta_1) + i\sin(\theta_1)$ and $z_2 = \cos(\theta_)) + i\sin(\theta_2)$. Then:

$$\begin{aligned}\rho(z_1)\rho(z_2) &= \begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) \\ -\sin(\theta_1) & \cos(\theta_1) \end{pmatrix} \begin{pmatrix} \cos(\theta_2) & \sin(\theta_2) \\ -\sin(\theta_2) & \cos(\theta_2) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) \\ -\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix} = \rho(z_1 z_2).\end{aligned}$$

It is easy to check that $\rho(1) = Id_2$ and that this map is not only a homomorphism as shown above, but also an isomorphism due to the unique association of a singular value of θ to one element of S^1 and $SO_2(\mathbb{R})$ each.

$\square$

Now, we will attempt to construct a Haar measure for $G = SO_2(\mathbb{R})$. Let $\mu := \frac{\mathrm{d}\theta}{2\pi}$, it is easy to check that it is a Haar measure for G when each element of G is associated with a unique value of θ.

$$\mu(gA) = \int_{gA} \frac{\mathrm{d}\theta}{2\pi} = \frac{1}{2\pi}[(g + A_f) - (g + A_i)] = \frac{1}{2\pi}(A_f - A_i) = \int_A \frac{\mathrm{d}\theta}{2\pi} = \mu(A)$$

$$\mu(G) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} = 1$$

Then, applying definition 3.1, we have:

$$M_4(SO_2(\mathbb{R}), GL_n(\mathbb{R})) = \int_G (Tr\ \rho(g))^4 \mathrm{d}\mu(g)$$
$$= \int_0^{2\pi} (2\cos\theta)^4 \frac{\mathrm{d}\theta}{2\pi} = \frac{8}{\pi}\int_0^{2\pi} (\cos\theta)^4 d\theta = 6.$$

We have calculated the fourth moment of $SO_2(\mathbb{R}) \subset GL_2(\mathbb{C})$ by its definition using calculus. This method can be generalised to calculate the fourth moment of $SO_n(\mathbb{R}) \subset GL_n(\mathbb{C})$ for larger n. In Section 4.5 however, we will calculate the latter using representation theory.

4.2 Fourth moment of S_4 acting on $\mathbb{R}^3$

There exists an isomorphism between S_4 and the group of matrices describing the map of 4 different points to themselves on 3d space for all sets of these 4 points in which no permutation is congruent to another.

Now, let V be a $k-$vector space and B, B' be bases of V. Let Q be the transition matrix of $B \to B'$. Let $\phi : V \to V$ be a linear map. Let A_B and $A_{B'}$be the representation of ϕ under B and B'. Then, since the trace of matrices can be regarded as an abelian group: $Tr(A_{B'}) = Tr(QA_BQ^{-1}) = Tr(QQ^{-1}A_B) = Tr(A_B)$, meaning that the trace of all conjugates are the same. This serves as the motivation for our lemma:

Lemma 4.2 *Let $\sigma, \sigma' \in S_n$ where S_n is the group of permutations order n. Then σ is conjugate to σ' if and only if they share the same type, defined as the set of lengths of the disjoint cycles the permutation can be partitioned into.*

Proof Let us first examine the truth of this statement if $sigma$ is only a single cycle. Let $\eta \in S_n$ and $(k_1, ..., k_r)$ be a cycle in S_n. Let $x \in \{\eta(k_1), ...\eta(k_n)\}$, say $x = \eta(k_i)$. Then $\eta(k_1, ..., k_r)\eta^{-1}$ can be seen as a map of x.

$$\eta(k_1, ..., k_r)\eta^{-1}(x) = \eta(k_1, ..., k_r)\eta^{-1}(\eta(k_i)) = \eta(k_{i+1})$$

Therefore, as permutations within S_n, $\eta(k_1, ..., k_r)\eta^{-1} = (\eta(k_1), ...\eta(k_r))$, which preserves the length of the cycle. Now let $\sigma \in S_n$, $\sigma = \mu_1, ..\mu_r$ be the disjoint product of cycles. Then:

$$\eta\sigma\eta^{-1} = \eta(\mu_1, ..\mu_r)\eta^{-1} = \prod_{i=1}^{r} \eta\mu_i\eta^{-1}$$

From our earlier result, we know that $\eta\mu_i\eta^{-1}$ preserves the length of the cycle μ_i and therefore, $\eta\sigma\eta^{-1}$ preserves the type of σ as a whole. This proves both results in the lemma.

□

Using the lemma and our earlier result, we can choose a representative tetrahedron and a representative permutation matrix from each type to calculate M_4. The tetrahedron of choice has a face parallel to the xy-place, a rotational axis to that face coinciding with the z-axis, and an axis of reflection on the xz plane. Then:

Type of Permutation	of permutations	a representative	Trace
1-cycle	1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	3
2-cycle	$\binom{4}{2} = 6$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	1
3-cycle	$2 \times \binom{4}{3} = 8$	$\begin{pmatrix} \cos(\frac{2\pi}{3}) & \sin(\frac{2\pi}{3}) & 0 \\ -\sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	0
4-cycle	$3! = 6$	?	?
2 2-cycles	$\binom{4}{2} \times \frac{1}{2} = 3$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	-1

All of the above examples are easily verifiable, so no space will be spent here explaining them. Notably however, there was no quick example for permutations of cycle length 4. However, we can find out the unknown trace of permutation matrices of cycle length 4 by referring to the First orthogonality Relation. This serves as motivation for another lemma:

Lemma 4.3 *Let $G = S_4$ where S_4 is the group of permutations of order 4 and $V = \mathbb{C}^3$. Then the group action of G on V is irreducible.*

Proof We note that $\rho : S_4 \to GL(V)$ describes the rotation or reflection of one particular tetrahedron to itself. Let W be a non-trivial vector space that currently simply consists of all points on $\overleftrightarrow{WO}$, where O is the origin and W is a random point. For simplicity's sake, let one of the vertexes of this tetrahedron be on the origin. Then, the application of S_4 to the origin vertex of the tetrahedron would reach the origin and 2 other points, creating 3 non-parallel lines as vector spaces. Since each point on all 3 lines are a part of vector space W, $W = \mathbb{C}^3 = V$. Hence, by contradiction, group action G on V is irreducible.

$\square$

Using the above lemma, we can apply the First Orthogonality Relation to this problem:

$$\langle \chi_V, \chi_V \rangle = \frac{1}{|S_4|} \sum_{g \in S_4} (\chi_v(g))^2 = dim(End(V)^G)$$

By Schur, since there exists and isomorphism between irreducible representation, $End(V)^G = \mathbb{C}Id$ and therefore $dim(End(V)^G) = 1$. Since we already have traces of the representations of all different types of permutations except 4-cycles, let us now let x be the trace of 4-cycles and apply it to the above equation:

$$\frac{1}{24}\left(3^2 + \binom{4}{2} \cdot 1^2 + 0 + 3! \cdot x^2 + \binom{4}{2} \cdot \frac{1}{2} \cdot 1^2\right) = 1$$

We get $x^2 = 1$. Now, putting everything back into the formula for fourth moments, we have:

$$M_4(S_4, GL_3(R)) = \frac{1}{24} \sum_{g \in S_4} |\chi_v(g)|^4 = \frac{1}{24}(81 + 6 + 0 + 6 + 3) = 4$$

4.3 Fourth moment of S^n acting on $\mathbb{C}^n$

We have a natural action of S_n on $\mathbb{C}^n$ defined by: for $\sigma \in S_n$ and $x = (x_1, ..., x_n) \in \mathbb{C}^n$, $\sigma(x) := (x_{\sigma(1)}, ..., x_{\sigma(n)})$. This action gives an injection of groups: $S_n \hookrightarrow GL_n(\mathbb{C})$. In this section we will calculate the fourth moment of this inclusion.

Let us first begin with several observations:

1. There exists an isomorphism between the S_n and the group of $n \times n$ permutation matrices.

2. The entries equal to 1 along the diagonal of a permutation matrix represent those elements which are mapped back to themselves.

Let X be defined as the set of elements upon which S_n acts. Then the trace of any permutation matrix of degree n is equivalent to $|\sigma|$ where $\{\sigma \in S^n,\ x_i \in X : \sigma x_i = x_i\}$. We note that $\forall k \in \mathbb{N}$, k^4 represents the total number of ways 4 non-identical objects could be placed into $|\sigma|$ containers, each container capable of holding multiple objects. Therefore, for each element $g \in S_n$, the value $Tr(\rho(g))^4$ is equivalent to the number of ways we can select 4 not necessarily different elements which are mapped back to themselves in this g. Hence, the value $\sum_{g \in S_n} Tr(\phi(g))^4$ represents the number of times in all permutations where every quartet of elements x_a, x_b, x_c, x_d (a, b, c, d not necessarily different from each other) is mapped back to their original positions. Let us now divide this counting problem into cases:

Case 1 ($a \neq b \neq c \neq d$): $\binom{n}{4} \times (n-4)! \times 4! = n!$
Case 2 ($a = b = c = d$): $\binom{n}{1} \times (n-1)! \times 1! = n!$
Case 3 ($a = b \neq c = d$): $\binom{n}{2} \times (n-2)! \times \binom{4}{2} = 3n!$
Case 4 ($a = b = c \neq d$): $\binom{n}{2} \times (n-2)! \times \binom{4}{3} \times 2! = 4n!$
Case 5 ($a = b \neq c \neq d$): $\binom{n}{3} \times (n-3)! \times \binom{4}{2}\binom{3}{1} \cdot 2 = 6n!$
Summing the 5 cases, we find that $\sum_{g \in S_n} Tr(\rho(g))^4 = 15n!$. Then, we have:

$$M_4(S_n, GL_n(\mathbb{C})) = \frac{1}{S_n} \sum_{g \in S_n} Tr(\rho(g))^4 = \frac{1}{n!} \times 15n! = 15$$

4.4 Fourth moments of finite subgroups of $SO(3)$ acting on $\mathbb{R}^3$.

The classification of finite subgroups of $SO(3)$ is a basic and well-known result [1]. In this part, we will calculate the fourth moments of finite subgroups of $SO(3)$. To this end, we will recall the classical method used to classify finite subgroups of $SO(3)$ following Artin's textbook [1]. The idea is to determine the action of finite groups on regular polygons. The action gives a representation of the finite group in question, and by calculating explicitly the character table, we can calculate the fourth moments. We will exploit the first orthogonality relation (Section 2.4) to calculate these fourth moments.

Before examining the proposition, we need to define a few terms and prove several lemmas. Assume G acts on X and $x \in X$, then:

Definition 4.1 The orbit of x is defined as $O_x := \{gx : g \in G\}$.

Definition 4.2 The stabiliser of x is define as $Stab_G(x) := \{g \in G : gx = x\}$.

Lemma 4.4 $|G| = |O_x||Stab_G(x)|$.

Proof An intuitive explanation is that for all $x \in X$, all elements $g \in G$ can be divided into two sets based on the effect of their action on x: whether gx results in a change in the element ($gx \in O_x$), or it does not($g \in Stab_G(x)$). Then to construct any element of G, we need to choose one element from either set (Id belongs to both sets), proving the lemma.

$\square$

Remark 4.1 For all $A \in O(3)$ and $x, y \in \mathbb{R}^3$, $\langle Ax, Ay\rangle = \langle x, y\rangle$.

Proof

$$\langle Ax, Ay\rangle = (Ax)^\intercal(Ay) = (x^\intercal A^\intercal)(Ay) = x^\intercal(A^\intercal A)y = x^\intercal y = \langle x, y\rangle.$$

$\square$

Let $\overrightarrow{x} \in S^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, then by the above remark, we have $|A\overrightarrow{x}|^2 = \langle Ax, Ax\rangle = \langle x, x\rangle = |\overrightarrow{x}|^2 = 1$. Hence, for any element $\overrightarrow{x} \in S^2$, we have $A\overrightarrow{x} \in S^2$. This creates the motivation for us to define a group action of $SO(3) \subset O(3)$ on the sphere of radius 1, S^2.

Definition 4.3 Let $G \subset SO(3)$ be a finite subgroup. The set of poles of G is defined as the set of fixed points $P_g \subset S^2$ such that there exists $g \in G$ where $gx = x$.

Remark 4.2 Any non-identity element in SO(3) has exactly 2 poles.

Lemma 4.5 *G acts on P_g i.e. $\forall g, x \in P_G$, $gx \in P_G$.*

Proof Let $g \in G$. Let $x \in P_G$, x must be a pole of some element $h \neq g\neg e \in G$ i.e $hx = x$. Then for $ghg^{-1} \in G$, $ghg^{-1}(gx) = g(hx) = gx$. Hence gx is a pole for ghg^{-1}, a non identity element of G. □

Let $G \subset SO(3)$ be a finite subgroup, then we know that P_G is a finite set. Now for the group action G on P_G, let $|G| = N$, Let $O_1, O_2, ...O_k$ be the orbits of this group action. Let $|O_i| = r_i$, let $n_i = |Stab_G(x_i)|$ for $x_i \in O_i$.

Lemma 4.6

$$2(N-1) = \sum_{i=1}^{k}(n_i - 1)r_i.$$

Proof Let $Q := \{(g, x) \in G \times P_G : g \neq e, gx = x\}$. Then there are two ways of counting $|Q|$:

Method 1: For any given $g \in G$, $g \neq e$ there are exactly 2 elements $x_1, x_2 \in P_G$ satisfying $gx_1 = x_1$ and $gx_2 = x_2$. Hence $|Q| = 2(N-1)$.

Method 2: For any given $x \in P_G$, the set $g \in G$ s.t. $gx = x$ and $g \neq e$ is just: $Stab_G(x) - \{e\}$. Hence, to count the total number of pairs of Q, we need to count the total number of pairs of elements from selected from each orbit and corresponding stabilizer: $|Q| = \sum_{i=1}^{r}(n_i - 1)r_i$.

□

By the lemma, we have:

$$2(N-1) = \sum_{i=1}^{k}(n_i - 1)r_i,$$

$$2 - \frac{2}{N} = \sum_{i=1}^{k} 1 - \frac{1}{n_i} \quad (*)$$

Now, let us apply some casework onto this problem:

Let $N = 1$, then clearly $G = e$, a subgroup all to itself. For this case, the fourth moment is $1 \times (3)^4 = 81$.

Let $N > 1$. Suppose $k = 1$, then: $2 - \frac{2}{N} = 1 - \frac{1}{n_1} \to \frac{1}{n_1} = \frac{2}{N} - 1 > 0$ which is clearly impossible. Then, we realize there must be more than one orbit.

Let $N > 1$ and $k = 2$. Rearranging $(*)$, we have $\frac{1}{n_1} + \frac{1}{n_2} = \frac{2}{N} \to n_1 = n_2 = N$. Then, we know that each orbit contains one element and that the stabilizer for both orbits is the entire group. This implies that the elements of G are rotations with respect to axis P_1P_2, the two elements comprising the two orbits.

Claim 4.1 Let $G \subset SO(3)$ be a finite subgroup containing only rotations with respect to an axis P_1P_2, then there exists an element $g_{min} \in G$ such that for all $g \in G$, $g = g_{min}^k$ for some $k \in \mathbb{Z}^*$.

Proof Since G is finite, there exists an element $g_{min} \neq 0 \in G$ whose angle of rotation θ_{min} is minimal. Then any other angle rotation in G is a positive integer multiple of θ. If not, say there exists some $g \in G$ whose angle of rotation θ is not a multiple θ_{min}, then for some group member $g_{min}^{-k}g$, the associated angle of rotation $0 < \theta - k\theta_{min} < \theta_{min}$, which contradicts the original choice of g_{min}. $\square$

Hence, we have shown all finite cyclic group of elements are subsets $C_n \subset SO(3)$. Hence, we will try to calculate the fourth moment of these groups:

let $g_{min} = \begin{pmatrix} \cos(\frac{2\pi}{n}) & \sin(\frac{2\pi}{n}) & 0 \\ -\sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for some $n \in \mathbb{Z}^+$, then the rest of the elements of G can be expressed as $g_{min}^k = \begin{pmatrix} \cos(\frac{2k\pi}{n}) & \sin(\frac{2k\pi}{n}) & 0 \\ -\sin(\frac{2k\pi}{n}) & \cos(\frac{2k\pi}{n}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $k \in \{1, 2, ..., n-1\}$. Then:

$$M_4(C_n, GL_3(\mathbb{R})) = \frac{1}{n}\sum_{i=0}^{n-1}\left(2\cos(\frac{2k\pi}{n})+1\right)^4 = \frac{1}{n}\sum_{i=0}^{n-1}(e^{\frac{2ki\pi}{n}}+e^{-\frac{2ki\pi}{n}}+1)^4$$

$$= \frac{1}{n}\sum_{i=0}^{n-1} e^{\frac{8ki\pi}{n}}+e^{-\frac{8ki\pi}{n}}+4(e^{\frac{6ki\pi}{n}}+e^{-\frac{6ki\pi}{n}})+10(e^{\frac{4ki\pi}{n}}+e^{-\frac{4ki\pi}{n}})+16(e^{\frac{2ki\pi}{n}}+e^{-\frac{2ki\pi}{n}})+19.$$

The above form is written out in the form of a sum of multiples of 4 geometric series of common ratio $e^{\pm\frac{2ki\pi}{n}}$, $e^{\pm\frac{4ki\pi}{n}}$, $e^{\pm\frac{6ki\pi}{n}}$, and $e^{\pm\frac{8ki\pi}{n}}$ and 19. We note that each of the sum of the above series when their common ratio is not 1 is equal to 0. Hence, we can divide into cases:

Case 1 ($n = 2$): When $n = 2$, $e^{\pm\frac{4ki\pi}{n}} = e^{\pm\frac{8ki\pi}{n}} = 1$ and hence, $M_4(C_2, GL_3(\mathbb{R})) = 1 + 1 + 10 + 10 + 19 = 41$;

Case 2 ($n = 3$): When $n = 3$, $e^{\pm\frac{6ki\pi}{n}} = 1$ and hence, $M_4(C_3, GL_3(\mathbb{R})) = 4 + 4 + 19 = 27$;

Case 3 ($n = 4$): When $n = 4$, $e^{\pm\frac{8ki\pi}{n}} = 1$ and hence, $M_4(C_4, GL_3(\mathbb{R})) = 1 + 1 + 19 = 21$;

Case 4 ($n > 4$): When $n > 4$, all of the common ratios are equal to 0 and hence, $M_4(C_n, GL_3(\mathbb{R})) = 19$.

Returning to the analysis of the subgroups of $SO(3)$, we move on to the case where there are 3 orbits:

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1 + \frac{2}{N}.$$

Case 1: Let $n_1 = 2$, $n_2 = 2$, $n_3 = k \geq 2 \Rightarrow N = 2k, r_3 = 2$. Hence, for O_3, there are 2 poles $\{P, P'\}$ making up the orbit. Every element $g \in G$ either fixes them, in which case they are rotations of multiples of $\frac{2\pi}{n}$ about $\overline{PP'}$, or interchanges them, in which case they are reflections across some axis perpendicular to $\overline{PP'}$. Therefore, these subgroups of $SO(3)$ are simply dihedral groups D_{2k} in $3D$, the group of symmetries of a regular k-gon in $3D$ space. Then, let us construct an example of D_{2k}. Let G_1, rotations of multiples of $\frac{2\pi}{N}$ about the z-axis and G_2 be the reflection across the x-axis. To construct the elements of D_{2k}, we must choose an element from G_1 and choose whether or not to apply the reflection. However, since the reflection across the x axis is $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and the elements of $g \in G$ can be expressed in the form $g = \begin{pmatrix} \cos(\frac{2j\pi}{N}) & \sin(\frac{2j\pi}{N}) & 0 \\ -\sin(\frac{2j\pi}{N}) & \cos(\frac{2j\pi}{N}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for all $j \in \{0, 1, ..., n-1\}$, the application of the reflection to g would result in a trace of 1 for all g. Then the fourth moment of D_{2k} can be calculated as:

$$M_4(D_{2k}, GL_n(\mathbb{R})) = \frac{1}{2k}\sum_{j=1}^{k}\left(2\cos(\frac{2j\pi}{N}) + 1\right)^4 + 1^4.$$

Referring to the work on cyclic groups above, we have:

Subcase 1 ($k = 2$): $M_4(D_{2k}, GL_n(\mathbb{R})) = 21$;
Subcase 2 ($k = 3$): $M_4(D_{2k}, GL_n(\mathbb{R})) = 14$;
Subcase 3 ($k = 4$): $M_4(D_{2k}, GL_n(\mathbb{R})) = 11$;
Subcase 4 ($k > 4$): $M_4(D_{2k}, GL_n(\mathbb{R})) = 10$.

Case 2: Let $n_1 = 2$, $n_2 = 3$, $n_3 = 3 \Rightarrow N = 12$. The poles of orbit O_2 and O_3 are the vertices of a regular tetrahedron, meaning that this $G \subset SO(3)$ is the subgroup fixing these vertices. Hence, the subgroup displayed here is simply the A_4 (the group of permutations of order 4 which have sign 1). Referring to our previous work in subsection 4.2, we have:

$$M_4(A_4, SO(3)) = \frac{1}{12}(1 \times 81 + 3 \times 1) = 7.$$

Case 3: Let $n_1 = 2$, $n_2 = 3$, $n_3 = 4 \Rightarrow N = 24$. The poles of orbit O_1, O_2, O_3 are the sides, vertices, and faces of a cube. Examining O_2, we ascertain that this $G \subset SO(3)$ is the subgroup fixing the vertices of this cube. However, since reflection and rotation do not count, we treat opposite vertices of the cube as the same object. This motivates us to create a group action S_4 on the 4 pairs of vertices. Noting lemma 4.2, we create a representative chart to determine the trace of the various matrices:

Type of Permutation	of permutations	a representative	Trace
1-cycle	1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	3
2-cycle	$\binom{4}{2} = 6$	?	x
3-cycle	$2 \times \binom{4}{3} = 8$	$\begin{pmatrix} \cos(\frac{2\pi}{3}) & \sin(\frac{2\pi}{3}) & 0 \\ -\sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	0
4-cycle	$3! = 6$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	1
2 2-cycles	$\binom{4}{2} \times \frac{1}{2} = 3$	?	y

Using a similar proof of lemma 4.3, we can ascertain that this group action is irreducible and hence, we can apply the first orthogonality theorem to solve for the value of x and y. Let V be the above representation and W be the trivial representation, then:

$$\langle \chi_V, \chi_W \rangle = \frac{1}{24}(1 \times 3 + 6x + 3y + 6 \times 1) = 0 \Rightarrow 2x + y = -3,$$

$$\langle \chi_V, \chi_V \rangle = \frac{1}{24}(1 \times 3^2 + 6x^2 + 3y^2 + 6 \times 1^2) = 1 \Rightarrow 2x^2 + y^2 = 3.$$

Solving these two equations, we have $x, y = -1$. Therefore:

$$M_4(S_4, SO(3)) = \frac{1}{24}(1 \times 3^4 + 6 \times (-1)^4 + 6 \times (1)^4 + 3 \times (-1)^4) = 4.$$

Case 4: Let $n_1 = 2$, $n_2 = 3$, $n_3 = 5 \Rightarrow N = 60$. The poles of O_2 are the vertices of a regular dodecahedron. We ascertain that this $G \subset SO(3)$ is the subgroup fixing the vertices of this dodecahedron. However, since rotation and reflection do not count, we can assign 5 different elements to the vertices of each pentagonal face, and treat those 5 sets as the same object. This motivates us to assign a representation of A_5 to the set of vertices. Using a very similar approach to the one outlined in Case 3 and subsection 4.2, we create a representative chart to determine the trace of the various:

Type of Permutation	# of permutations	Representative	Trace
1-cycle	1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	3
3-cycle	$2 \times \binom{5}{3} = 20$	$\begin{pmatrix} \cos(\frac{2\pi}{3}) & \sin(\frac{2\pi}{3}) & 0 \\ -\sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	0
2 2-cycles	$\frac{1}{2} \times \binom{5}{2} \times \binom{3}{2} = 15$	?	x
5-cycle (1,2,3,4,5)	$\frac{4!}{2}$	$\begin{pmatrix} \cos(\frac{2\pi}{5}) & \sin(\frac{2\pi}{5}) & 0 \\ -\sin(\frac{2\pi}{5}) & \cos(\frac{2\pi}{5}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\frac{1+\sqrt{5}}{2}$
5-cycles (1,3,5,2,4)	$\frac{4!}{2}$	$\begin{pmatrix} \cos(\frac{4\pi}{5}) & \sin(\frac{4\pi}{5}) & 0 \\ -\sin(\frac{4\pi}{5}) & \cos(\frac{4\pi}{5}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\frac{1-\sqrt{5}}{2}$

Then, using a similar proof as used in lemma 4.3, we can ascertain that this representation is irreducible and hence, we can apply the first orthogonality theorem to solve for the value of x. We get $x = -1$ and hence:

$$M_4(A_5, GL_3(\mathbb{R})) = \frac{1}{60}\left(3^4 + 15 + 12\left(\frac{1+\sqrt{5}}{2}\right)^4 + 12\left(\frac{1-\sqrt{5}}{2}\right)^4\right) = 3.$$

4.5 Fourth moment of $SO(n) \subset O(n) \subset GL_n(\mathbb{C})$

Proposition 4.1 $M_4(O(n), GL_n(\mathbb{R})) = 3$.

Proof By representation theory, let $V = \mathbb{C}^n$ and $SO(n)$ and $O(n)$ act on V. Then, by Proposition 3.1, we have:

$$M_4(O(n), GL_n(\mathbb{C})) = \dim(End(End(V)))^{O(n)}$$

$$M_4(SO(n), GL_n(\mathbb{C})) = \dim(End(End(V)))^{SO(n)}$$

This motivates us to create irreducible representations of $End(V)$ as $O(n)$ representations. Let: $End^0(V) := \{A \in M_n(\mathbb{C}) : Tr(A) = 0\}$ $Sym^0(V) := \{A \in M_n(\mathbb{C}) : A = A^\intercal\}$. $Anti(V) := \{A \in M_n(\mathbb{C}) : A = -A^\intercal\}$.We note that $T_e(O(n)) \otimes_{\mathbb{R}} \mathbb{C} = Anti(V)$. We can prove this using a very similar proof to the first part of Example 2.7.

We now have irreducible representations of $End(V)$ as $O(n)$ representations:

$$\begin{aligned} End(V) &= \mathbb{C}Id \oplus End^0(V) \\ &= \mathbb{C}Id \oplus Sym^0(V) \oplus Anti(V) \\ &= \mathbb{C}Id \oplus Sym^0(V) \oplus Lie(O(n)_{\mathbb{C}}). \end{aligned} \tag{4}$$

Lemma 4.7 *$\mathbb{C}Id$, $Sym^0(V)$, and $Lie(O(n)_{\mathbb{C}}$ are distinct, irreducible representations of $O(n)$.*

Proof

The proof of $\mathbb{C}Id$ being irreducible is trivial.

We refer to the Exercise V.2 of [5] for the proof of $Sym^0(V)$ being irreducible.

We refer to [2] for the proof of $Lie(O(n)_{\mathbb{C}}$ being irreducible

□

Then, by Remark 3.1, we have $M_4(O(n), GL_n(\mathbb{C})) = 1^2 + 1^2 + 1^2 = 3$.

We note that this result coincides with the results calculated in [7].

□

Theorem 5 *Let $n \geq 2$ be an positive integer:*

$$M_4(\mathrm{SO}(n, \mathbb{C}), GL_n(\mathbb{C}) = \begin{cases} 6, & \textit{if } n = 2 \\ 4, & \textit{if } n = 4 \\ 3, & \textit{if } n \neq 2, 4 \end{cases}$$

Proof

When $n = 2$, by [5] we have $Sym^0(V) = W_1 \oplus W_2$ where W_1 and W_2 are two distinct $1D$ representations distinct from $\mathbb{C}$Id and Lie$(O(n)) = \mathbb{C}Id$. Therefore, for $n = 2$, $End(V) = 2\mathbb{C}Id \oplus W_1 \oplus W_2$. Then, by Remark 3.1, we have $M_4(SO(n), GL_n(\mathbb{C})) = 2^2 + 1^2 + 1^2 = 6$. Note that this coincides with our direct calculation in Section 4.1.

When $n = 4$, by [2], we have Lie$(O(n)) = V_1 \oplus V_2$, 2 distinct $3D$ irreducible subrepresentations. Therefore, for $n = 4$, we have $End(V) = \mathbb{C}Id \oplus Sym^0(V) \oplus V_1 \oplus V_2 \Rightarrow M_4(SO(n), GL_n(\mathbb{C})) = 1^2 + 1^2 + 1^2 + 1^2 = 4$.

When $n \neq 2, 4$, $\mathbb{C}\mathrm{Id}$, $Sym^0(V)$, $\mathrm{Lie}(O(n)_{\mathbb{C}}$ are all distinct irreducible $SO(n)$-representations. Therefore: $End(V) = \mathbb{C}\mathrm{Id} \oplus Sym^0(V) \oplus \mathrm{Lie}(O(n)_{\mathbb{C}} \Rightarrow M_4(SO(n), GL_n(\mathbb{C})) = 1^2 + 1^2 + 1^2 = 3$

□

Corollary 4.1 *Let $O^-(n)$ be defined as the set of $n \times n$ orthogonal matrices whose determinant is -1. Let μ_1 be the Haar measure on $O^-(n)$ induced from the Haar measure of $O(n)$. Then:*

$$\int_{O(n)^-} |\mathrm{Tr}g|^4 d\mu_1(g) = \begin{cases} 0, & \text{if } n = 2 \\ 1, & \text{if } n = 4 \\ 3/2, & \text{if } n \neq 2, 4 \end{cases} . \tag{5}$$

Proof
Let μ_2 be the Haar measure of $SO(n)$ acting on $\mathbb{R}^n$. By Proposition 4.1, we have:

$$\begin{aligned} 3 = M_4(O(n), GL_n(\mathbb{C})) &= \int_{O(n)} |\mathrm{Tr}(g)|^4 d\mu_1(g) \\ &= \int_{SO(n)} |\mathrm{Tr}(g)|^4 d\mu_1(g) + \int_{O(n)} |\mathrm{Tr}(g)|^4 d\mu_1(g) \\ &= \frac{1}{2} \int_{SO(n)} |\mathrm{Tr}(g)|^4 d\mu_2(g) + \int_{O(n)} |\mathrm{Tr}(g)|^4 d\mu_1(g) \end{aligned} \tag{6}$$

Then, by applying Theorem 5 to the above result, we prove the corollary.

□

References

[1] M. Artin, *Algebra*, Pearson Prentice Hall, 2011.

[2] Humphreys, James. *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics 9, Springer-Verlag, 1972.

[3] Kang, David. (2011). *Group Representations and Character Theory.*

[4] Katz, Nicholas. (2004). *Larsen's Alternative, Moments, and the Monodromy of Lefschetz Pencils*, to Joe Shalika on his 60'th birthday.

[5] A. W. Knapp, *Lie Groups Beyond an Introduction*, Progress in Mathematics(Birkhäuser), 1996.

[6] E. Kowalski, *An Introduction to the Representation Theory of Groups*, Graduate Studies in Mathematics 155, AMS, 2010.

[7] L. Pastur, V. Vasilchuk, *On the Moments of Traces of Matrices of Classical Groups*, Comm. in Math. Phy. 252, 149-166, 2004.

Acknowledgements

I would like to thank Professor Lei Song at the School of Mathematics at the Sun Yat-sen University for his invaluable advice throughout the research process. I would also like to express my gratitude to my parents and math teachers for nurturing my interest in mathematics from a young age, especially in the field of algebra which this paper would explore. Last but not least, I would like to thank the Committee of Yau Awards for this wonderful opportunity to do research in mathematics.