Analysis of Kidney Transplant System Using Markov Process Models

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Summary

Abstract: We use Markov processes to develop a mathematical model for the U.S. kidney transplant system. We use both mathematical models and computer simulations to analyze the effect of certain parameters on transplant waitlist size and investigate the effects of policy changes on the model's behavior.

Our results show that the waitlist size is increasing due to the flooding of new waitlist members and insufficient deceased donor and living donor transplants available. Possible policy changes to improve the situation include presumed consent, tightening qualifications for joining the waitlist, and relaxing the requirements for accepting deceased donors.

We also evaluate alternative models from other countries that would reduce the waitlist, and examine the benefits and costs of these models compared with the current U.S. model. We analyze kidney paired exchange along with generic n-cycle kidney exchange, and use our original U.S. model to evaluate the benefits of incorporating kidney exchange.

We develop a model explaining the decisions that potential recipients face concerning organ transplant, then expand this consumer decision theory model to explain the decisions that potential organ donors face when deciding whether to donate a kidney.

We finally consider an extreme policy change—the marketing of kidneys for kidney transplants—as a method of increasing the live-donor pool to reduce waitlist size.

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Introduction

The American organ transplant system is in trouble: Waitlist size is increasing ; as of February 2007, 94,000 candidates were waiting for a transplant, among them 68,000 waiting for kidneys. We create a mathematical model using a Markov process to examine the effects of parameters on waitlist size and to investigate the effects of policy changes. Possible policy changes to improve the situation include assuming that all people are organ donors unless specifically specified (presumed consent), tightening qualifications for joining the waitlist, and relaxing the requirements for accepting deceased donors.

We evaluate alternative models from other countries that could reduce the waitlist, and examine the benefits and costs of these models compared with the current U.S. model. We analyze the Korean kidney paired exchange along with the generic n-cycle kidney exchange, and use our original U.S. model to evaluate the benefits of incorporating the kidney exchange. The Korean model increases the incoming rate of live donors, which is preferable because live-donor transplants lead to higher life expectancy. However, this policy alone cannot reverse the trend in waitlist size.

We also develop a model explaining the decisions that potential recipients face concerning organ transplant. We expand this consumer decision theory model to explain the decisions that potential organ donors face when deciding whether or not to donate a kidney. Finally, we consider an extreme policy change-the marketing of kidneys for kidney transplants as a method of increasing the live donor pool to reduce waitlist size. We consider two economic models: one in which the government buys organs from willing donors and offsets the price via a tax, and one in which private firms are allowed to buy organs from donors and offer transplants to consumers at the market-equilibrium price.

Task 1: The U.S. Kidney Transplant System

Background: Kidney Transplants

- **Blood Type**: Recipient and donor must have compatible blood types (**Table 1**).

- **HLA**: Recipient and donor must have few mismatches in the HLA antigen locus. Because of diverse allelic variation, perfect matches are rare. Mismatches can cause rejection of the organ.

- **PRA**: PRA is a blood test that measures rejection to human antibodies in the body. The value is between 0 and 99, and its numerical value indicates the percent of the U.S. population that the blood's antibodies reacts with. High PRA patients have lower success rates among potential donors[U so it is more difficult to locate donate matches for them (**Table 2**).

Table 1.
Compatible blood types [American National Red Cross 2006].

Recipient blood type	Donor red blood cells must be:							
AB+	O−	O+	A−	A+	B−	B+	AB−	AB+
AB−	O−		A−		B−		AB−	
A+	O−	O+	A−	A+				
A−	O−		A−					
B+	O−	O+			B−	B+		
B−	O−				B−			
O+	O−	O+						
O−	O−							
In U.S. population:	7%	38%	6%	34%	2%	9%	1%	3%

Table 2.
Relationship between PRA and transplant waiting time [University of Maryland . . . 2007].

Peak PRA	Proportion of waiting list	Median waiting time to transplant (days)
0–19	60%	490
20–79	21%	1,042
80+	19%	2,322

Explanation of Model

The Organ Procurement and Transplantation Network's (OPTN) priority system for assigning and allocating kidneys is used as the core model for the current U.S. transplantation system [Organ Procurement . . . 2006]. The OPTN kidney network is divided into three levels: the local level, the regional level, and the national level. There are 270 individual transplant centers distributed throughout the U.S. [Dept. of Health and Human Services 2007], organized into 11 geographic regions.

The priority system for allocation of deceased-donor kidneys to candidates on the waitlist takes into account proximity of recipient to donor, recipient wait time, and match to donor, with location carrying greater weight, according to a point system [Organ Procurement . . . 2006]:

- **Wait time points** A candidate receives one point for each year on the waiting list. A candidate also an additional fraction of a point based on rank on the list: With n candidates on the list, the rth-longest-waiting candidate gets $1 - (r-1)/n$ points. So, for example, the longest-waiting candidate ($r = 1$) gets one additional point, the newest arrival on the list ($r = n$) gets $1/n$ additional points.

- **Age points** The young receive preferential treatment because their expected lifetime with the transplant is greater. Children below 11 years of age get 4 additional points, and those between 11 and 18 get 3 additional points.

- **HLA mismatch points** Because there are two chromosomes, the possible number m of mismatches in the donor-recipient (DR) locus of the HLA sequence is 0, 1, or 2. A candidate-donor pair gets $2 - m$ points.

Model Setup

We model the entry and exit of candidates from the waitlist with a continuous-time Markov birth/death process [Ross 2002]. It accommodates reduction of the waitlist size (arrivals of living donors and deceased donors and deaths and recoveries of waitlist candidate) and waitlist additions.

- In 2006, 29,824 patients were added to the kidney transplant waitlist, while 5,914 transplants had living donors, so $5914/(29824 + 5914) \approx 17\%$ of incoming patient cases have a willing compatible living donor.
- The procedure for allocating deceased-donor kidneys is [Organ Procurement ... 2006, 3.5, 3–16ff]:
  - First, match the donor blood type with compatible recipient blood types. The only exceptions are:
    - Type O donors must be donated to type O recipients first, and:
    - Type B donors must be donated to type B recipients first.
  - Perfect matches (same blood type and no HLA mismatch) receive first priority.
  - If a kidney with blood type O or B has no perfect-matching candidates in the above procedure, then the pool is reopened for all candidates.
  - In the 17% of cases of no a perfect match with any recipient [Wikipedia 2007], then sort by PRA value (higher priority to high PRA; high PRA means low compatibility, which likely means being on the waitlist for a long time), then by regional location of the kidney, then by points in the point allocation system.

Summary of Markov Process

Let N_t be a random variable indicating the number of people in the waitlist at time t. The properties of N_t can be generalized in **Figure 1**, where

- Each arrow represents a possible event at the current state (N).
- The rate at which each event occurs is exponentially distributed.

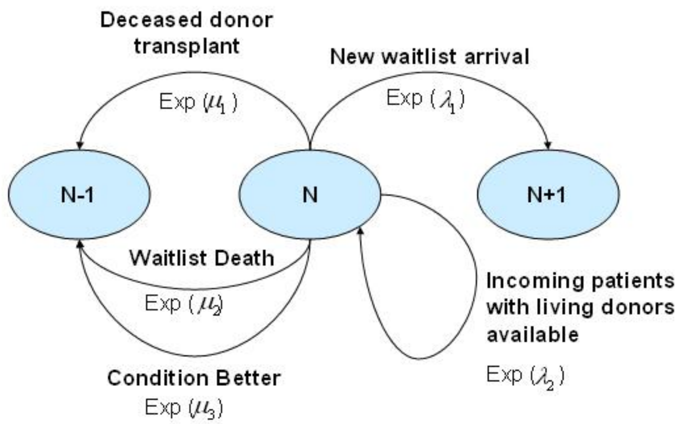

Figure 1. Markov process model of waitlist.

- After the event occurs, by memorylessness of the exponential distribution, the time is reset to zero, as if nothing has happened.
- Wait time is assumed zero for compatible live donor transplants.
- Because there are so many local centers (270), we simplify our model to consider the region (of which there are 11) as the lowest level of waitlist candidates.
- Candidates who become medically unfit surgery are removed from the wait list and in our model are classified as deaths.
- Candidates whose conditions improve enough are removed from the waitlist. Both these people and those recovering from surgery have exponential remaining lifetime with mean 15 years.
- We use the parameter values in **Table 3**, which come from the OPTN database using values from 2006.

Table 3.
Means of exponential distributions.

Symbol	Rate	Mean
λ_1	new waitlist arrivals	81.7 d
λ_2	incoming patients with living donors available	16.2 d
$\lambda_3 = \lambda_1 + \lambda_2$	total incoming patients (independent RVs)	81.7 + 16.2 = 97.9 d
μ_1	arrivals of deceased donor transplants	26.9 d [Norman 2005]
μ_2	waitlist deaths	27.0 d
μ_3	waitlist condition improves per day	2.4 d
$\mu_4 = \mu_1 + \mu_2 + \mu_3$	waitlist departures (independent RVs)	26.9 + 27.0 + 2.4 = 56.3 d
T_{AB}	time of life after surgery [European Medical Tourism 2007]	0, if candidate dies; 15 y with transplant.

Analysis of Model

Our two variables to indicate strength of model strategy are the number of people in the waitlist (or the number of people who get transplants) and optimizing the matches so as to maximize lifetime after receiving a transplant.

Efficient Allocation of Kidney Transplants

We build a new model to take into account the effects of both distance and optimal match. A kidney arriving at a center can be given to the best matching candidate at that center, the best in the region, or the best in the country.

Of 10,000 candidate recipients, on average 37 are from the center, 873 are from the region outside the center, and 9,090 are from the nation outside the region. Using a uniform distribution on $(0, 1)$, we randomly assign scores to each of the 10,000, rank them by score, and take the highest rank at each level. We iterate this process 10,000 times and find the average rank of the top candidate in each area (**Table 4**).

Table 4.
Average quality of top candidate in each area.

	Probability that top candidate is in this group			Average rank (from bottom) of top candidate among 10,000
Center	$\frac{1}{270}$	$=$	0.37%	9739.7
Region outside center	$\frac{1}{11} - \frac{1}{270}$	$=$	8.72%	9989.7
Nation outside region	$\frac{10}{11}$	$=$	90.90%	9999.9

Transportation of the kidney can lead to damage, because of time delay in transplanting. Thus, we posit a damage function f that depends on the location of the recipient: lower in the center, slightly higher in the region but outside the center, and even higher in the country but outside the region, i.e.,

$$f(\text{local}) < f(\text{regional}) < f(\text{national}).$$

Let us assume that when a kidney arrives in a center, it goes to the center, the region, or outside the region with probabilities a_1, a_2, and a_3. Let G be the weighted score for the kidney, with

$$\begin{aligned} G \quad = \quad & a_1 \cdot \big(1 - f(\text{local})\big) \cdot \text{score}_{\text{local}} + a_2 \cdot \big(1 - f(\text{regional})\big) \cdot \text{score}_{\text{regional}} \\ & + a_3 \cdot \big(1 - f(\text{national})\big) \cdot \text{score}_{\text{national}}. \end{aligned} \qquad \textbf{(1)}$$

and expected value

$$\begin{aligned} E(G) \quad = \quad & a_1 \cdot \big(1 - f(\text{local})\big) \cdot 9739.7 + a_2 \cdot \big(1 - f(\text{regional})\big) \cdot 9989.7 \\ & + a_3 \cdot \big(1 - f(\text{national})\big) \cdot 9999.9. \end{aligned} \qquad \textbf{(2)}$$

Optimizing G as a function of the a_i is a linear programming problem, but we cannot solve it without assessing the damage function for different regions.

Minimizing the Waitlist

There are some who argue that the wait time assignment is too lax and leads to unfair waitlists. In the current system, urgency is specifically stated as have no effect on the points used to determine who receives a transplant [Organ Procurement ... 2006]. A patient is permitted to join the waitlist (in more than one region, even) when kidney filtration rate falls below a particular value or when dialysis begins. Getting on the waitlist as early as possible helps "pad" the points for waiting time. A patient not yet on dialysis can afford to wait longer yet may receive a kidney sooner than others joining later who have more urgent need. Urgency has no effect on a patient's rank for receiving a kidney. A possible solution is to tighten the conditions for joining the waitlist, so that that a patient's wait time begins at dialysis. This policy would slow the rate of growth of the waitlist, at the expense of more waitlisted patients dying.

A strategy to increase the rate of deceased-donor arrival, already policy in Illinois, is to presume that everyone desires to be an organ donor unless they specifically opt out.

Figure 2 shows the field space of combinations from rates for these two policies.

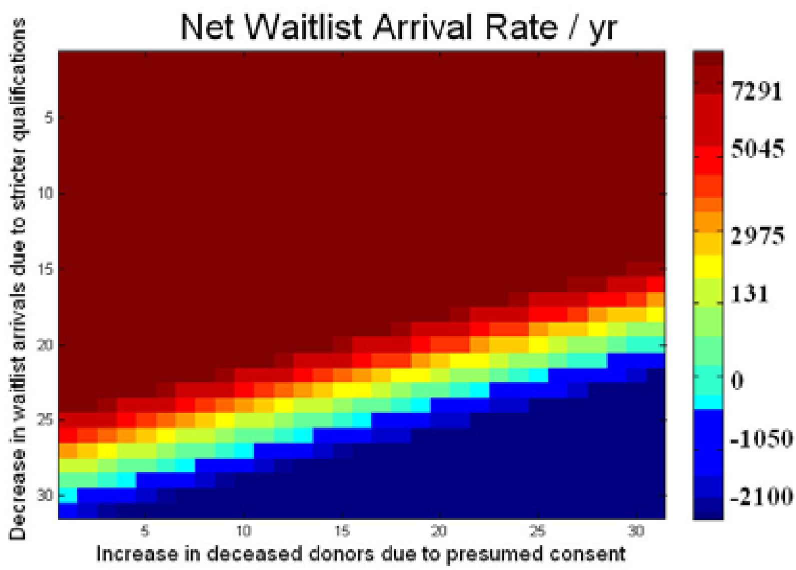

Figure 2. Net waitlist arrival rate per year.

Using both strategies could make net waitlist arrival rate negative, for example, if waitlist arrivals can be decreased by 25% and donor size by 17%.

Model Strengths

- The Markov process, with exponentially distributed entry / exit times, makes calculations simple.

- Minimizing the waitlist depends on only two variables.
- The model incorporates HLA values, PRA distributions, no-mismatch probabilities, region distribution, and blood-type distribution and compatibility requirements.
- The model is compatible with alternative strategies, such as a paired exchange system.

Model Weaknesses

- Remaining lifetime after surgery should be adjusted, since an exponential distribution for remaining lifetime is appropriate only until a certain age.
- The model cannot account for patients'. We assume that all patients offered a kidney take it if the HLA value is reasonable, which may not be the case.
- The model does not make distinctions for race and socioeconomic status. Different races have differing wait times [Norman 2005, 457].
- We assume independence of random variables, so that increasing or decreasing parameters will not affect other parameters.
- Our emphasis on waitlist size neglects waitlist time; another approach would be to try to minimize waitlist wait time.

Tasks 2 and 3: Kidney Paired Exchange

Background

As noted at the University of Chicago Hospitals, "In 10 to 20 percent of cases at the Hospitals, patients who need a kidney transplant have family or friends who agree to donate, but the willing donor is found to be biologically unsuited for that specific recipient" [Physicians propose ... 1997].

In the simple kidney paired exchange system (**Figure 3**), there are two pairs of patient / donor candidates. Each donor is incompatible with the intended patient but compatible with the other patient. Surgery is performed simultaneously in the same hospital on four people, with two kidney removals and two kidney transplants.

However, for not all patient-donor pairs will there be a mutually compatible partner pair. In such a case, it is possible for the cycle to expand to n patient-donor pairs, with each donor giving to a compatible stranger patient (**Figure 4**). Since such an exchange requires at least $2n$ surgeons at the same hospital, higher-order exchanges are less desirable on logistic grounds.

A kidney paired exchange program does not affect the intrinsic model outlined for Task 1. The only change when live incompatible pairs get swapped is

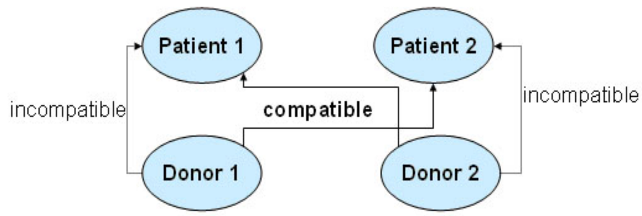

Figure 4. Kidney paired exchange system.

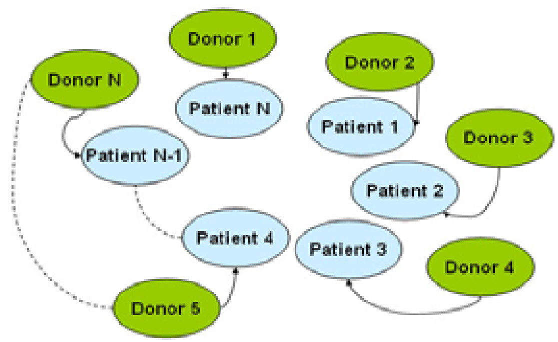

Figure 5. n-way kidney exchange.

that the rate of candidates entering the waitlist is reduced. However, for those in the waitlist, the same procedure is still being used.

- Kidney paired exchanges have higher priority than larger cycles, for logistical reasons.
- Recipients must receive a transplant in the region in which they are on the waitlist (this reduces travel time).
- Exchanges with no mismatch are prioritized over exchanges with mismatch.

Waiting time for an exchange is assumed to be 0, as was live donor matches in Task 1. After an exchange, all individuals involved are removed from the pools of donors and recipients.

We use Region 9 as a sample region to test our model. Region 9 has a waitlist (6058) similar to the average waitlist per region, and 909 candidates (15%) have willing but incompatible donors. We ran our simulation 100 times and computed averages. **Table 5** shows extrapolation of the results nationwide.

Analysis

In 2006, there were 26,689 kidney transplants nationwide, including only a few kidney paired exchanges. The approximately 9,656 additional transplants yearly indicated in **Table 5** would have been a 36% increase and would have reduce the waitlist correspondingly by 14%, from 69,983 to 60,327.

Table 5.
Averaged results of repeated simulations of multiple-pair transplant exchange nationwide (extrapolated from Region 9 data).

Kind of match	Transplants	Percentage
2-way no mismatch	2	
2-way non-perfect	9,646	
3-way no mismatch	0	
3-way non-perfect	8	
Total transplants	9,656	92%
Candidates with willing but incompatible donor	10,497	

Another option is to consider multiple exchanges for all donor-recipient pairs in a particular center. This minimizes the travel time required for the patients, while improving the computational power of the search algorithm. A center has on average 259 candidates, of whom 39 have willing but incompatible donors available. For this sample size, we get on average 25 transplants (65%), compared to 92% under exchange at the regional level. Furthermore, the proportion of high-quality transplants is also smaller. The benefits of a center-only exchange system are personal and psychological: Patients live close to the surgery location, which means better support from both family and familiar physicians.

Task 4: Patient Choice Theory

Suppose a patient is offered a barely compatible kidney from the cadaver queue. There are two options:

- take the bad-match kidney immediately, or
- wait for a better match,
  - from the cadaver queue or
  - from a paired exchange.

We consider two cases: without paired exchange and with it.

Model 1: Decision Scenario without Paired Exchange

Of transplants with poorly matched kidneys, 50% fail after 5–7 years. So we assume that the lifetime after a poorly matched kidney transplant is exponentially distributed with mean 6 years [Norman 2005, 458].

We translate data of **Table 6** on survival probabilities to exponential variables with mean λ by solving $P(\text{survive } t \text{ years}) = e^{-\lambda t}$.

Table 6.
Rates for patient survival [National Institute of Diabetes and Digestive and Kidney Diseases 2007].

Time (y)	Dialysis		Live-donor transplant		Cadaver transplant	
	p	λ	p	λ	p	λ
1	.774	0.256	.977	0.0233	.943	0.0587
2	.632	0.229	.959	0.0209	.907	0.0209
5	.315	0.231	.896	0.0220	.819	0.0399
10	.101	0.229	.753	0.0284	.591	0.0526
Avg.		**0.236**		**0.0237**		**0.0500**

We first diagram the wait strategy for the scenario of waiting on dialysis for a deceased-donor kidney, with no kidney paired exchange (**Figure 5**).

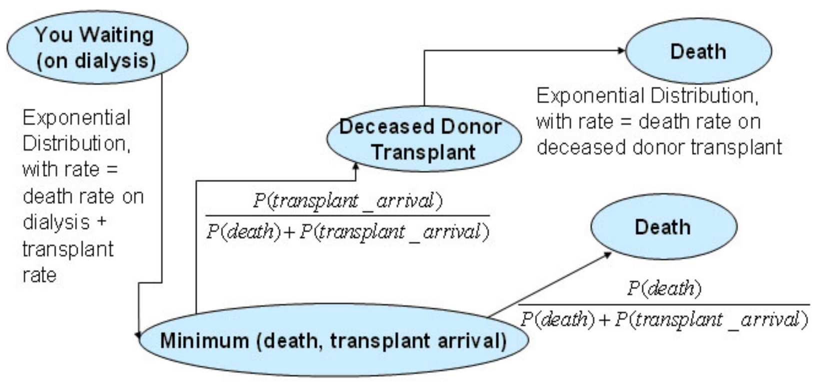

Figure 6. Wait strategy with no paired exchange.

We then calculate expected remaining lifetime with this strategy. We use

$$P(\text{deceased-donor transplant}) = \frac{\text{deceased-donor transplants}}{\text{waitlist}} = \frac{10659}{75711} = .140,$$

using 2006 data [Organ Procurement ... 2007][1]. We have

$$E(\text{lifetime}) = \frac{0.236}{0.236+0.140}\left(\frac{1}{0.236+0.140}\right) + \frac{0.140}{0.236+0.140}\left(\frac{1}{0.236+0.140}+\frac{1}{0.050}\right) \approx 10 \ \text{ years.}$$

If instead the patient chooses to undergo immediate transplant with a bad match deceased-donor kidney, then remaining lifetime is exponentially distributed with rate 0.167, so

$$E(\text{lifetime}) \approx 6.0 \ \text{ years.}$$

[1]Author note: In hindsight, a better measure is probably to use the **Table 2** median waiting times to calculate average waiting time (2.66 years). Using the assumption of exponential distribution, we have that probability of an arrival of a deceased-donor kidney in one year is $e^{(}-2.66) \approx .07$.

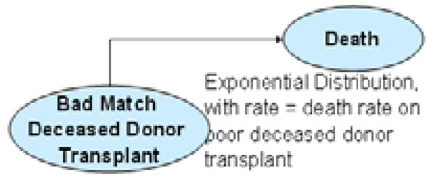

Figure 7. Transplant strategy with no paired exchange.

The expected remaining lifetime for the wait strategy is 4 years greater, so we recommend that strategy. It assumes that the patient is risk-neutral. Being on dialysis leads to an expected remaining lifetime of 4.2 years, which is less than the expected remaining lifetime for a bad-match kidney. The decision hinges on how much risk the patient is willing to take.

Model 2: Decision Scenario with Paired Exchange

This modified scenario leads to **Figure 7**. Since between 10% and 20% of patients have willing but incompatible donors [Physicians propose ... 1997], and lacking any better data, we use .15 as the probability of a kidney paired exchange being possible. Using similar calculations as before, we find the expected remaining lifetime for the wait strategy:

$$E(\text{lifetime}) \approx 19.5 \text{ years.}$$

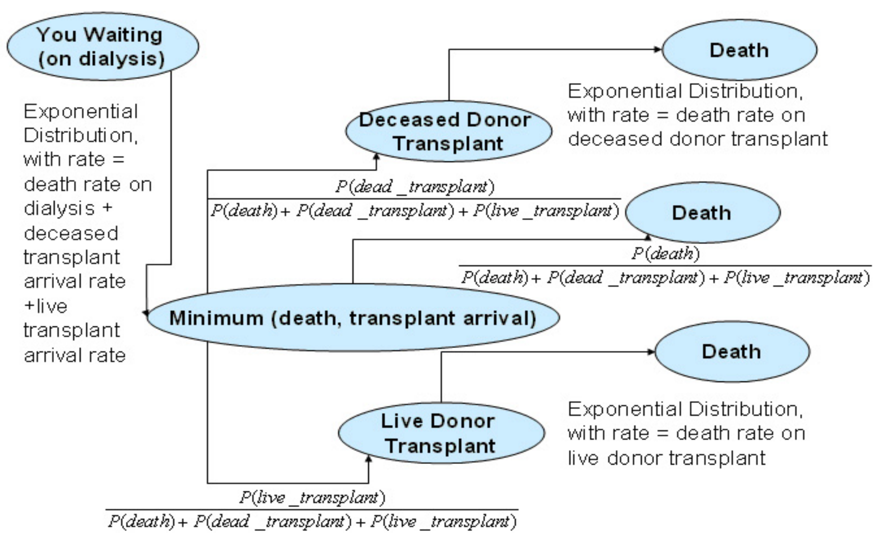

Figure 8. Wait strategy with paired exchange.

Model Strengths

- The model compares strategies numerically.

- Values for survival rate, rate of death, and rate of donor arrival are available from data, and these parameters can be easily adjusted for different scenarios.
- The model can be modified to accommodate other strategies, new categorizations of transplants (perhaps divide transplants into grades A, AA, AAA for quality).

Task 5: New Organ Market

Another method to increase the rate of incoming live donors is to implement a market allowing people to sell organs for transplantation. Currently, it is illegal in the U.S. to "transfer an organ for valuable consideration" [National Organ Transplant Act 1984]. There are two possible ways a market can work: government-managed (**Figure 8**) or using a public market (**Figure 9**).

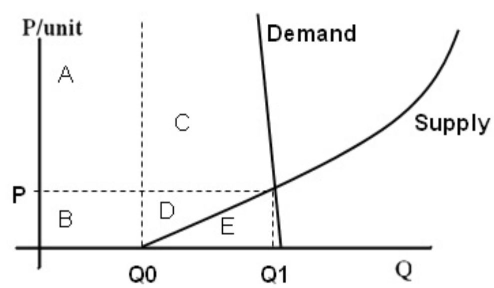

Figure 9. Government-managed organ-buying system.

Originally, Q_0 units of organs were available (via cadaver and living donor transplants). In the government-managed model, people sell their kidneys to the government at the market equilibrium price. The government pays $D + E$ for the kidney transplants, so the economy suffers a tax of $D + E$. However, the customers gain the consumer surplus of C and the suppliers gain the supplier surplus of D, so benefit of having more kidneys available to society is $C + D - (D + E) = C - E$. Because of the inelasticity of demand for kidney transplants, $C > E$, so $C - E > 0$. Therefore, a government-managed system would eliminate the waitlist because Q_1 would likely be greater than the total number of people on the waitlist. However, government-managed systems are known to be slow and inefficient [Krauss 2006], so people in this market would have long wait times for a transplant. The increase from Q_0 to Q_1 would be drastic, leading to a strain on hospitals and on the health care system.

A possible solution to the long wait times intrinsic to government-managed surgeries is to privatize the market and allow private companies to buy kidneys and sell surgeries. The Q_0 donors who originally donated for free would still be in the model (but we assume that they would still be uncompensated), so

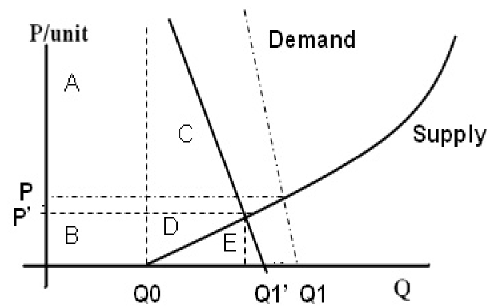

Figure 10. Free market for organs.

the companies buy organs only from the remaining supply curve. The market equilibrium is still the same coordinate, but this time the consumer surplus is increased by C and the supplier surplus by D, so the benefit to society is $C + D$ instead of $C - E$, and the free-market system is more efficient to society by $D + E$. Because the company's demand curve is less inelastic than the customer's demand curve, it is likely that companies will buy fewer kidneys, at a lower price, than the government would. However, companies may take advantage of the inelastic demand curves and try to make long-run profits.

A free-market system has benefits over the government-managed system; it is more efficient, and it would lead to more transplants if the government-managed system had longer wait times and inefficient allocations due to the bureaucratic difficulties of managing a nationwide kidney industry.

However, the free-market system also has disadvantages. Matches involving different races are less likely to lead to good transplants, because tissue-type gene sequences have different distributions by race and hence lower likelihood of compatibility across races. Race-associated differences in genetics could lead to race-specific markets (and prices) for kidneys.

Another possible protest against this market structure is ethical dilemmas regarding the selling of organs. The organ is a part of one's body, but questions arise whether one "owns" one's organs. One religious view would be that because the body is sacred, it would be wrong to sell one's body for monetary gain. Furthermore, introduction of a market for kidneys could lead some companies to try to start stemcell or nonliving organ farms, a dangerous step according to the realm of bioethics.

While a kidney market would increase efficiency and lead to more transplants, both the government-sponsored and the free-market versions could each provide major ethical problems.

Task 6: Potential Donor Decision Theory

We calculate the probability that donors will donate for various situations. We build a model similar to the patient's decision scenario, only now we consider the states of the world from a potential donor's viewpoint. We consider three cases: kidney donation to a loved one, to a random person, and to a random person in a kidney exchange system so that a loved one can receive a kidney transplant as well.

We first evaluate the case when the kidney could be donated to a loved one.

Donor Decision: Donate to a Loved One?

Strategy 1: Donate

Let C be a score assigned to the strategy when you donate a kidney to a loved one. Let C be a function of three random variables X, Y and Z, where

- X is the remaining lifetime of the recipient given a live donor transplant,
- Y is the remaining lifetime of the donor given a live donor transplant, and
- Z is the pain and depression value of the donor after donating a kidney.

The remaining lifetime of a recipient of a live-donor transplant is exponentially distributed with rate 0.0237. We also know that the perioperative mortality rate is 3 deaths per 10,000 donors (0.03%), and that 2% of donors encounter major complications [Najarian et al. 1992]. (Some donors experience depression or conflict with family members, but these problems are unrelated to the success of the transplantation [Liounis et al. 1988].) Thus, X, Y, and Z are as follows:

- X is exponentially distributed with rate $\lambda = 0.0237$;
- $Y = \begin{cases} 0, & \text{with probability } 0.03\%, \\ T_{\mathrm{N}}, & \text{with probability } 97.97\%, \\ T_{\mathrm{MC}}, & \text{with probability } 2\%, \end{cases}$

  where T_{N} is the random variable for remaining lifetime of a normal person, and T_{MC} is the random variable for remaining lifetime of a person with major complications of a donor from kidney transplant;
- Z is the numerical value for amount of depression, conflict, and anger that results from donating kidney.

Hence C from this example is given by:

$$C = a_1 X + a_2 Y - a_3 Z,$$

where a_1, a_2, and a_3 are weights for how important each variable is. These weights reflect the emphasis on each variable by the given donor.

$$\begin{aligned} E(C) &= a_1 E(X) + a_2 E(Y) - a_3 E(Z) \\ &= a_1 \cdot 42.19 + a_2 \cdot (0.0003 \cdot 0 + 0.9797 \cdot \overline{T_{\mathrm{N}}} + 0.02 \cdot \overline{T_{\mathrm{MC}}}) - a_3 \cdot \overline{Z}. \end{aligned}$$

Strategy 2: Don't Donate.

In this case, we notice several changes to the variables X, Y, and Z,

- There are two scenarios: Your loved one does not get a transplant and dies, or your loved one receives a transplant. In the first case, the time until your loved one dies is exponentially distributed with rate 0.236; and in the second case, the time until your loved one receives a transplant is exponentially distributed with rate 0.140 (same as in Task 4). Thus, the minimum of the two, which is the time until the first event occurs, is exponentially distributed with rate $(0.140 + 0.236) = 0.376$. Thus, we have

$$X = \min\{T_{\text{die}}, T_{\text{transplant}}\} \cdot P(\text{die}) \\ + \left(\min\{T_{\text{die}}, T_{\text{transplant}}\} + T_{\text{RLADS}}\right) \cdot P(\text{transplant}),$$

  where T_{RLADS} is the remaining life after transplant from a deceased donor, which is exponentially distributed with mean 0.050. We know that $E(X) \approx 10.11$ years.

- $Y = T_n$.
- $Z = 0$.

It follows again from $C = a_1X + a_2Y - a_3Z$ that

$$E(C) = a_1E(X) + a_2E(Y) - a_3E(Z) = a_1 \cdot 10.11 + a_2 \cdot \overline{T_N} - a_3 \cdot 0.$$

The expected value of the benefit of the transplant strategy over the no-transplant strategy is

$$E\big[C(\text{transplant})\big] - E\big[C(\text{no transplant})\big] = \\ a_1 \cdot 32.08 + a_2 \cdot (-0.0203 \cdot \overline{T_{\text{N}}} + 0.02 \cdot \overline{T_{\text{MC}}}) - a_3 \cdot \bar{Z}.$$

The first component is positive, while the second and third are negative, since $\overline{T_{MC}} < \overline{T_N}$. The result can be either negative or positive, depending on the weights a_i.

Donor Decision: Donate to an Unknown Person?

In this case, the model stays exactly the same but the values of the a_i will be different.

Donor Decision: Donate via Paired Exchange?

We consider an N-pair exchange. We continue using the system provided in the previous example but must include more parameters:

- X_1 is the remaining lifetime of the loved-one recipient given a live donor transplant, and is exponentially distributed with rate $\lambda = 0.0237$;
- X_2 is the remaining lifetime of the $N-1$ stranger recipients who are each given a live donor transplant, and obviously $X_2 = (N-1)X_1$;
- Y is the remaining lifetime of the donor given a live donor transplant, same as in non-paired-exchange scenario; and
- Z is the pain and depression value of the donor after donating a kidney.

We then have

$$\begin{aligned} E(C) &= a_1 E(X_1) + a_2(N-1)E(X_2) + a_3 E(Y) - a_4 E(Z) \\ &= a_1 \cdot 42.19 + a_2 \cdot (N-1) \cdot 42.19 \\ &\quad + a_3 \cdot \big(.0003 \cdot 0 + 0.9797 \cdot \overline{T_N} + 0.02 \cdot \overline{T_{MC}} - a_4 \cdot \overline{Z}. \end{aligned}$$

For the no-transplant strategy, then in this case instead of one recipient being forced to wait for a donor, all N recipients must now wait, since no size $N-1$ cycle exists. Thus, we have:

- X_1 is the same as in the original X for the no-transplant, no-exchange strategy, but now transplants arrive faster because new live transplants are available. So

  $$\begin{aligned} X_1 = {} & \min\{T_{\text{die}}, T_{\text{DEADtransplant}}, T_{\text{LIVEtransplant}}\} \cdot P(\text{die}) \\ & + (\min\{T_{\text{die}}, T_{\text{DEADtransplant}}, T_{\text{LIVEtransplant}}\} + T_{\text{RLADS}}) \cdot P(\text{DEADtransplant}) \\ & + (\min\{T_{\text{die}}, T_{\text{DEADtransplant}}, T_{\text{LIVEtransplant}}\} + T_{\text{RLALS}}) \cdot P(\text{LIVEtransplant}), \end{aligned}$$

  where T_{RLAS} is the remaining life after surgery given that it is from a deceased donor. We know this to be exponentially distributed with mean 0.050, and $E(X_1) = 19.26$ years. This value is different from the no-exchange system because people on the waitlist will have a higher chance of receiving a transplant when the policy changes to permit and encourage exchanges.

- $X_2 = (N-1)X_1$ is the remaining lifetime of the $N-1$ stranger recipients who are each given a live donor transplant, and

  $$E(X_2) = (N-1)E(X_1) = (N-1) \cdot 19.26 \text{ years.}$$

- $Y = T_n$.
- $Z = 0$.

Hence we obtain

$$\begin{aligned} E(C) &= a_1 E(X_1) + a_2(N-1)E(X_2) + a_3 E(Y) - a_4 E(Z) \\ &= a_1 \cdot 19.26 + a_2 \cdot (N-1) \cdot 19.26 + a_3 \cdot \overline{T_N} - a_4 \cdot 0. \end{aligned}$$

Using both of the C values for transplant and no-transplant possibilities, we see that the A variable for choosing the transplant is: The expected value of the benefit of the transplant strategy over the no-transplant strategy is

$$E\big[C(\text{transplant})\big] - E\big[C(\text{no transplant})\big] =$$
$$a_1 \cdot 22.93 + a_2 \cdot (N-1) \cdot 22.93 + a_3 \cdot (-0.0203 \cdot \overline{T_N} + 0.02 \cdot \overline{T_{MC}}) - a_4 \cdot \bar{Z}.$$

We compare this value with $a_1 \cdot 32.08 + a_2 \cdot (-0.0203 \cdot \overline{T_N} + 0.02 \cdot \overline{T_{MC}}) - a_3 \cdot \bar{Z}$ for a no-exchange system.

The expected lifetime of the related patient has lower impact in the exchange model. This is because the related patient may receive exchange transplants in the future if you do not donate your organ through an exchange. While the effect of a_1 decreases, a new variable $a_2 \cdot (N-1) \cdot 42.19$ increases the probability that a donor decides to donate a kidney. This is because the donor feels responsible for increasing the lifetime for all N recipients in the size-N transplant exchange, because without that donor, none of the transplants would be possible. However, because the donor feels less attached to random recipients, we have $a_1 >> a_2$.

Model Strengths

The model

- provides a numerical value useful in gauging the probability that a donor decides to donate;
- is adjustable to any new system created;
- incorporates personal and psychological factors.

Model Weaknesses

Some variables and parameters are not independent, but our model assumes that the rates are independent.

Conclusions

After developing a model to understand the effects of components of the kidney transplant model, we have developed a list of solutions to the waitlist dilemma:

- **Tighten Waitlist Entry Requirement** Currently, patients join the waitlist when kidney filtration rate falls below a particular value or when dialysis begins. We recommend that only those whose conditions are at dialysis or worse should be allowed to join the waitlist. This change would lead to reduced inflow of waitlist candidates, dramatically improving the system.

- **Presumed Consent** Currently, those who wish to donate kidneys after death must have explicit documentation on hand when their bodies are retrieved. A new policy would assume that all deceased people are eligible for deceased-donor transplant, unless explicitly expressed otherwise. This change would dramatically increase the inflow of deceased donors.
- **Kidney Paired Exchange System** Many waitlist candidates have potential donors who cannot donate due to incompatible blood types or HLA. A kidney paired exchange system would match these people in a broad regional pool, identifying when donors can donate to the respective other paired recipient. This reduces the flow of incoming waitlist candidates.
- **Market Kidneys** We investigated government-sanctioned kidney purchases and a free market for kidneys. In both cases, the size of the waitlist diminishes with the number of live kidneys sold. However, a government bureaucracy could not handle the number of kidney transplants, so waiting time would increase for some, at least at first. In a free market, biological factors of kidney transplants could lead to discriminatory prices. Thus, marketing of kidneys is discouraged on the basis of parity.

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