2015-2016 Math League Contests, Grades 6– 8

Second-Round, Jan - Feb 2016

Instructions:

1. This second-round contest consists of two parts. Part 1 is math questions. Part 2 is English essay.
2. This document contains 12 pages in total, including this page.
3. For all the questions below, login to your account at http://www.mathleague.cn/, and enter your answers. Answers written on this sheet or any other place will NOT be credited.
4. In Part 1, you are asked to read a math subject, Chances and Probabilities, and supplementary materials. Then you have 20 questions to work on. You will need to give precise, unambiguous answers to Questions 1-19.
5. Question 20 is Projects and Papers, which means you need to do your research and write a paper for it. There is no word limit on your paper, but it doesn't necessary mean the longer the better. The best paper is precise and succinct. Please don't feel frustrated at all if you can't write a paper, as the topic is very hard for a middle school student, even for teachers. Please don't get discouraged even if you can't finish all Questions 1-19, as they are not trivial questions and it requires a lot reading and thinking. Students who can work out a few questions should be commended.
6. The more questions you answered correctly, the more credit you will get. The problems are ordered by content, NOT DIFFICULTY. It is to your advantage to attempt problems from throughout the test.
7. You can seek help by reading books, searching the Internet, asking an expert, and etc. But you can't delegate this to someone else and turn in whatever he/she wrote for you. To make it clear, the purpose of the second-round contest is to test your ability to read and research. You need to be the one who understand the topics and solve the problems. You will be caught if it is not the case during the interview.
8. For Part 1, you can write in either English or Chinese.
9. In Part 2, you are asked to write an essay regarding to one topic. You have to write in English in Part 2.
10. If you have any questions regarding the contest, please contact us at once at INFO@LTHOUGHTS.COM
11. Submission of your answers:
    a) For all the questions below, login to your account at http://www.mathleague.cn/, and enter your answers. Answers written on this sheet or any other place will NOT be credited.
    b) You need to submit your answers no later than 12:00AM, Feb 7, 2016, Beijing Time. Later submission will not be accepted.

ALL ANSWERS MUST BE EXACT UNLESS SPECIFIED. PROBABILITY MUST BE EXPRESSED IN EITHER DECIMALS OR FRACTIONS.

Part 1 –Chances and Probabilities

The following is an excerpt from some math book, Chances and Probabilities, (see separate document).

Supplementary Materials,(see separate document).

For all the questions below, login to your account at http://www.mathleague.cn/ , and enter your answers. Answers written on this document will NOT be credited.

Question 1:(6 pts)

A student rolls a die and randomly picks an integer from 1 to 4. Let X be the outcome of the die, and Y be the number he picks. Let W = X + Y.

(a) (2 pts)What is the sample space of W?[Note: Please list your answers in ascending order. There will be 10 or fewer answers. If you have fewer than 10 answers, please leave the trailing spaces blank.]

(b) (2 pts) What is the size of the sample space of W?

(c) (2 pts) What is the probability assignment of the sample space of W? In other words, what is the probability of *each* event in W?[Note: Please list your answers in the ascending order of the W-values. i.e. first enter the probability for the smallest W-value, then enter the probability for the second smallest W-value, and so on.There will be 10 or fewer answers. If you have fewer than 10 answers, please leave the trailing spaces blank.]

Question 2: (8 pts)

The Multiplication Rule:

A computer password consists of four letters (A through Z) followed by a single digit (0 through 9). Assume that the passwords are not case sensitive (i.e., that an upper case letter is the same as a lowercase letter).

(a) (2 pts) How many different passwords are possible?

(b) (2 pts) How many different passwords end in 1?

(c) (2 pts) How many different passwords do not start with Z?

(d) (2 pts) How many different passwords have no Z's in them?

Question 3: (12 pts)

Permutation VS Combination:

(a) (2 pts) There are 6 different positions in an office. The employer is selecting among 9 people to fill in these spots. Each position is held by only one person. How many different choices does he have?

(b) (2 pts) There are 12 players joining the chess tournament. If each player must play each of the other 11 players once, how many games are there in the tournament?

(c) (2 pts) In a class of 20 students, 4 students are absent today. How many different combinations of absent students are possible?

(d) (2 pts) In a math contest, the top 3 scorers have rewards. The first, second, and third finishers will win a $300 gift-card, $200 gift-card, and $100 gift-card respectively. If there are 50 students participating in this contest, how many different ways can the winners be selected? (Assume that there is no tie for each of the top 3 places.)

(e) There are 10 athletes entered in an Olympic events.

(i) (2 pts) In how many ways can one pick the winners of the gold, silver, and bronze medals?

(ii) (2 pts) In how many ways can one pick the seven athletes who will not earn any medals?

Question 4: (12 pts)

Suppose that three events—A, B, and C—are defined on a sample space S. Use the union, intersection, and complement operations to represent each of the following events:

(a) (2 pts) None of the three events occur.

(b) (2 pts) All of the three events occur.

(c) (2 pts) Only event A occurs.

(d) (2 pts) Exactly one of the three events occur.

(e) (2 pts) Exactly two of the three events occur.

(f) (2 pts) At least one of the three events occur.

Note: For this question, please write your answer on file “middle-school-answersheet.doc”, downloadable together with this document at www.mathleague.cn, and submit file “middle-school-answersheet.doc” at www.mathleague.cn after you are done.

Question 5: (10 pts)

The following is the breakdown of 100 marbles.

	Blue	Clear	Green	
Large	5	15	2	
Medium	25	5	3	
Small	30	10	5	

(a) (2 pts) If you choose one marble at random, what is P(Blue ∪ Small)?

(b)(2 pts) If you choose one marble at random, what is P(Clear | Small)?

(c) (2 pts) If you choose four marbles with replacement, what is P(at least one is clear)? [Hint: Use the idea of the complement event.]

(d) (2 pts) If you choose 20 marbles with replacement, what is P(exactly 15 are small)? [Hint: Reread Example 15.24].[Note: Answers must be in decimal form, correctly rounded to 4 decimal places.]

(e) (2 pts) [Irrelevant to the marble problem]: A factory is manufacturing n identical products. If p is the probability that a randomly selected product is defective, what is the probability that exactly k products are defective?[Assume that n is a positive integer, $0 < p < 1$, and k is a nonnegative integer such that $k \leq n$. Write an expression that answers the question in terms of n, k, and p. Hint: This question has the same idea as Question 5(d).]

Note: For this question, 5(e), please write your answer on file "middle-school-answersheet.doc", downloadable together with this document at www.mathleague.cn, and submit file "middle-school-answersheet.doc" at www.mathleague.cn after you are done.

Question 6: (6 pts)

Solve each of the following:

(a) (2 pts) If $P(A) = \frac{1}{2}$, $P(A|B) = \frac{4}{7}$, and $P(B|A) = \frac{2}{5}$. What is $P(A \cup B)$?

(b) (2 pts) If $P(A) = 0.6$, $P(B) = 0.4$, and $P(A|B^C) = 0.4$. What is $P(A \cup B)$?

(c) (2 pts) If A, B, and C are independent events, $P(A) = 0.5$, $P(B) = 0.3$, and $P(C) = 0.4$. What is $P(A \cup (B \cap C^C))$?

Question 7: (3 pts)

Steve, Dan, Tom, Jerry, and three other boys are standing in a line to take a group picture. If Steve must stand next to Dan, and Tom must stand next to Jerry, how many ways are there to arrange these seven boys in a line?

Question 8: (3 pts)

A bucket has two white and one black marbles. You will continuously draw marbles from the bucket until you get a white, but if you draw a black you put it back with another white marble into the bucket. If we let X = number of draws, what is the formula for P(X = *k*)? [Assume that *k* is a positive integer.]

Note: For this question, please write your answer on file "middle-school-answersheet.doc", downloadable together with this document at www.mathleague.cn, and submit file "middle-school-answersheet.doc" at www.mathleague.cn after you are done.

Question 9: (3 pts)

Six 3-member families enter a raffle. The director selects 4 winners and each winner will receivean iPhone 500 as a prize. What is the probability that the 4 winners come from 4 different families?

Question 10: (5 pts)

Suppose a test for a particular disease is 95% accurate, and suppose that only 0.1% of the individuals have this disease. If the test indicates that a person has that disease,

(a) (3 pts) what is the probability that the person actually has the disease? [Note: Answers must be in decimal form, correctly rounded to 4 decimal places.]

(b) (2 pts) The answer to part (a) may be a little counterintuitive because it is much lower than you expect. Explain how this can happen.

Note: For this question, 10(b), please write your answer on file "middle-school-answersheet.doc", downloadable together with this document at www.mathleague.cn, and submit file "middle-school-answersheet.doc" at www.mathleague.cn after you are done.

Question 11: (11 pts)

More On Counting:

(a) (2 pts) In how many different ways can 10 people form a line?

(b) (3 pts) In how many different ways can 10 people be seated around a circular table? [Hint: The answer to (b) is much less than the answer to (a). There are many different ways in which the same circle of 10 people can be broken up to form a line. How many?]

(c) (3 pts) In how many different ways can five boys and five girls get in line so that boys and girls alternate (boy, girl, boy, . . ., or girl, boy, girl, . . .)?

(d) (3pts) In how many different ways can five boys and five girls sit around a circular table so that boys and girls alternate?

Question 12: (11 pts)

In this question, let the term "word" represent an arrangement of letters (they do not have to mean anything). How many different "words" can be formed using all the letters in

(a) (2 pts) the word PARSLEY?

(b) (3 pts) the word PEPPER? [Hint: This question is very different (and harder) than the one in (a). What is the difference and what should you do?]

(c) (3 pts) the word MISSISSIPPI?

(d) (3 pts) The number of 10-letter "words" formed by REASSESSES is the same as the number of x-letter "words" formed by REDUCTIONS. What is the value of x?

Question 13: (4 pts)

What is the largest 3-digit prime factor of $\binom{2000}{1000}$?[Note: You may refer to the prime table below.]

Prime Numbers between 1 and 1,000

	2	3	5	7	11	13	17	19	23
29	31	37	41	43	47	53	59	61	67
71	73	79	83	89	97	101	103	107	109
113	127	131	137	139	149	151	157	163	167
173	179	181	191	193	197	199	211	223	227
229	233	239	241	251	257	263	269	271	277
281	283	293	307	311	313	317	331	337	347
349	353	359	367	373	379	383	389	397	401
409	419	421	431	433	439	443	449	457	461
463	467	479	487	491	499	503	509	521	523
541	547	557	563	569	571	577	587	593	599
601	607	613	617	619	631	641	643	647	653
659	661	673	677	683	691	701	709	719	727
733	739	743	751	757	761	769	773	787	797
809	811	821	823	827	829	839	853	857	859
863	877	881	883	887	907	911	919	929	937
941	947	953	967	971	977	983	991	997	

Question 14: (4 pts)

In a takeout restaurant, there are 10 different food choices. A guest is given 4 boxes to take out the food he/she likes. Each box can only contain one type of food. The types of food in the boxes that the guest takes out are not necessarily different. How many ways are there to put the food into 4 boxes?

Question 15: (4 pts)

Reading from left to right, for how many integers greater than 9 is true that every digit after the first exceeds the digit it follows? [Note: As an example, one such integer is 24789.]

Question 16: (4 pts)

In a drawer Richard has 5 pairs of gloves. Each pair has a different color. On Day 1 Richard selects two individual gloves at random from the 10 gloves in the drawer. On Day 2Richard selects 2 of the remaining 8 gloves at random and on Day 3 two of the remaining 6 gloves at random. The probability that Day 3 is the first day Richard selects matching gloves is m/n, where m and n are relatively prime positive integers. What is$m+n$?

Question 17: (8 pts)

On each of the following, assume that $k \leq n$. Write a formula in terms of$n$ and k as your answer.

(a) (4 pts) In how many ways can I place n indistinguishable objects into k distinguishable boxes so that each box contains at least 1 object?

(b) (4 pts) In how many ways can John select k out of the first$n$ positive integers, disregarding the order in which these k integers are selected, so that no two of the selected integers are consecutive integers?

Note: For this question, please write your answer on file "middle-school-answersheet.doc", downloadable together with this document at www.mathleague.cn, and submit file "middle-school-answersheet.doc" at www.mathleague.cn after you are done.

Question 18: (4 pts)

If we roll a fair die 5 times and the outcomes are a, b, c, d, and e respectively, the probability that $a \leq b \leq c \leq d \leq e$is m/n, where m and n are relatively prime positive integers. What is$m+ n$?

Question 19: (12 pts)

In terms of combinatorics, the entries in the Pascal's Triangle can be written as follows:

Row						
0	$\binom{0}{0}=1$					
1	$\binom{1}{0}=1$	$\binom{1}{1}=1$				
2	$\binom{2}{0}=1$	$\binom{2}{1}=2$	$\binom{2}{2}=1$			
3	$\binom{3}{0}=1$	$\binom{3}{1}=3$	$\binom{3}{2}=3$	$\binom{3}{3}=1$		
4	$\binom{4}{0}=1$	$\binom{4}{1}=4$	$\binom{4}{2}=6$	$\binom{4}{3}=4$	$\binom{4}{4}=1$	
5	$\binom{5}{0}=1$	$\binom{5}{1}=5$	$\binom{5}{2}=10$	$\binom{5}{3}=10$	$\binom{5}{4}=5$	$\binom{5}{5}=1$

(a) (4 pts) As observed, the sum of the *n*th row is 2^n.Using the idea of sets, subsets, and elements, explain why $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$.
(Hint: If set A has *n* elements, how many subsets does A have? Why?)

(b) (3 pts) Expand each of the following.

$(x + y)^0$ (Assume that x + y is not 0.)

$(x + y)^1$

$(x + y)^2$

$(x + y)^3$

$(x + y)^4$

$(x + y)^5$

What is notable in the coefficients and exponents?

(c)(2 pts) What is the formula for expanding $(x + y)^n$?

(d) (3 pts) Withoutusing the concept of factorials or the Pascal's Triangle, prove the following statement:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

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Question 20: (20 pts, note: paper with exceptional quality can get up to 40pts.)

Probability Theory in Games

Games of chance, be they for religious purposes or for profit, can be traced all the way back to some of the earliest civilizations—the Babylonians, Assyrians, and ancient Egyptians were all known to play primitive dice games as well as board games of various kinds.

In this project you are to write a paper discussing the development of probability theory as it relates to the games that involve chance. One possible way to start this project is to select a game, describe what it is, and how the concept of probability is involved in the game.

Note: For this question, please write your answer on file "middle-school-answersheet.doc", downloadable together with this document at www.mathleague.cn, and submit file "middle-school-answersheet.doc" at www.mathleague.cn after you are done.

Part 2 Essay (75 pts)

Let's define the word "discipline" as follows:

A way of behaving that shows a willingness to obey rules or orders (*Webster's*).

Without discipline, there's no life at all.

--Katharine Hepburn

Do you agree or disagree with this bolded quotation? Plan and write an essay in which you develop your point of view on this issue. Support your position with reasoning and examples taken from your reading, studies, experience, or observations.

Hints: Here are some questions that help you to craft the essay. You do not have to answer them, but think of them as you write:

1. What are some examples of having discipline?
2. What will happen if one violates discipline in a particular circumstance? Give a specific example and explain your points.
3. Will too much discipline hurt freedom and innovation? When is discipline too much? How much discipline is necessary?
4. Will different people treat discipline differently? For example, will soldiers, students, scientists, politicians, and etc. treat discipline differently?
5. What is the role of discipline in civilized society?
6. In your school, teachers might emphasize as much discipline as possible, while students might want as little discipline as possible. Who are right? How to balance this?

Note: For this question, please write your answer on file "middle-school-answersheet.doc", downloadable together with this document at www.mathleague.cn, and submit file "middle-school-answersheet.doc" at www.mathleague.cn after you are done.