Supplementary Materials

1. Set Theory (accompanies with 15.4: Probability Spaces)

Symbol	Symbol Name	Meaning / definition	Example
Suppose that A = {3,7,9,14}, B = {9,14,28}.			
$\{\ \}$	set	a collection of elements	A, B
$\vert$	such that	so that	$C = \{x \mid x \in \mathbb{R}, x<0\}$
$\cap$	intersection	objects that belong to both set A and set B	$A \cap B = \{9,14\}$
$\cup$	union	objects that belong to either set A or set B	$A \cup B = \{3,7,9,14,28\}$
$\subset$	subset	subset has fewer elements or equal to the set	$\{5,7\} \subset \{1,3,5,7\}$
$\not\subset$	not subset	left set not a subset of right set	$\{5,6\} \not\subset \{1,3,5,7\}$
$=$	equality	both sets have the same members	$\{5,6,7,8\} = \{5,6,7,8\}$
A^c	complement	all the objects that do not belong to set A	
$\in$	element of	Is a member of the set	$3 \in \{3,9,14\}$
$\notin$	not element of	Is not a member of the set	$1 \notin \{3,9,14\}$
Ø	empty set	a set that has 0 elements	{}

Note: The elements of the set do not necessarily have to be numbers.

For example, {a, b, c, d, e, f} is a set too.

As you read Section 15.4, you will find out about this.

Theorems:

1. If $A \subset B$ is true, then $A \cap B = A$ and $A \cup B = B$.

2. If $A = B$, then both $A \subset B$ and $B \subset A$ are true.

Ex:

Suppose there is sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and A, B, and C are subsets of S:

$A = \{1, 2, 4, 8\}$

$B = \{1, 3, 5, 7\}$

$C = \{2, 4, 6, 8\}$

Then:

$A \cap C = \{2, 4, 8\}$

$B \cap C = \{\} = \varnothing$

$A \cup B = \{1, 2, 3, 4, 5, 7, 8\}$

$B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8\} = S$

$A^C = \{3, 5, 6, 7\}$

$B^C = \{2, 4, 6, 8\} = C$

2. Probability Definition (see also 15.5—Equiprobable Spaces)

If all the elements in the sample space S are equally likely, then

$$P(A) = \frac{\#\, of\ elements\ in\ A}{\#\, of\ elements\ in\ S}.$$

3. Probability Theories (recap of 15.4—Probability Spaces. More specifically, Page 568)

Let A and B be the subsets of the sample space S. A probability function P is one that satisfies:

[Note: To denote the probability that event A will happen, the book uses Pr(A). On here and the problem set, we use P(A).]

1. $0 \leq P(A) \leq 1$.

2. $P(S) = 1$.

3. In general, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. (3.1)

   If A and B are mutually exclusive (which means $A \cap B = \emptyset$),

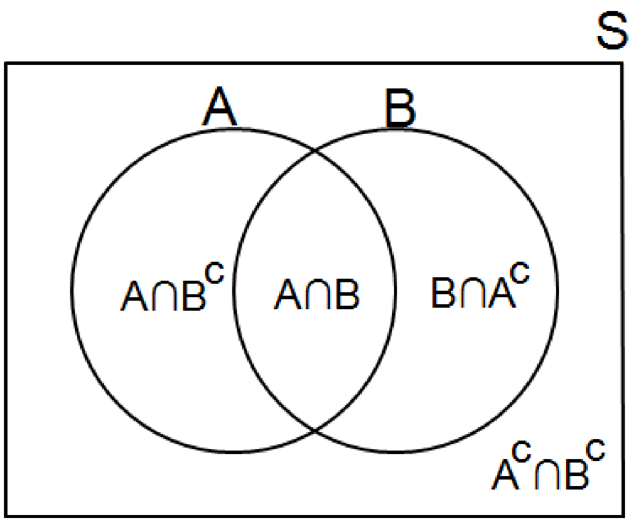
   then $P(A \cup B) = P(A) + P(B)$.

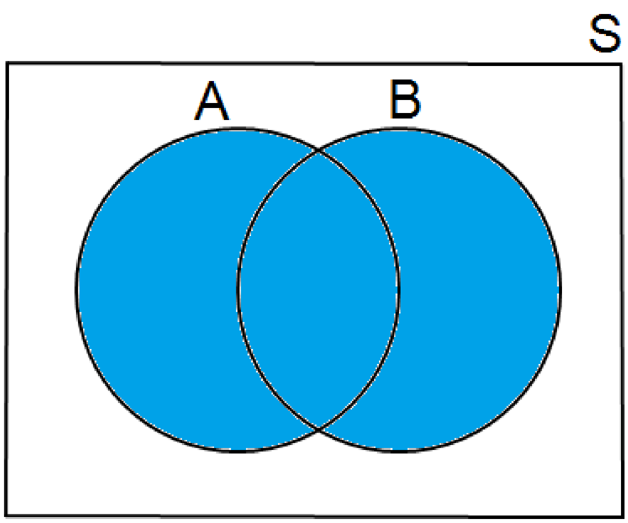
4. $A \cap A^C = \emptyset$ (A and A^C are mutually exclusive events.)

   $A \cup A^C = S$

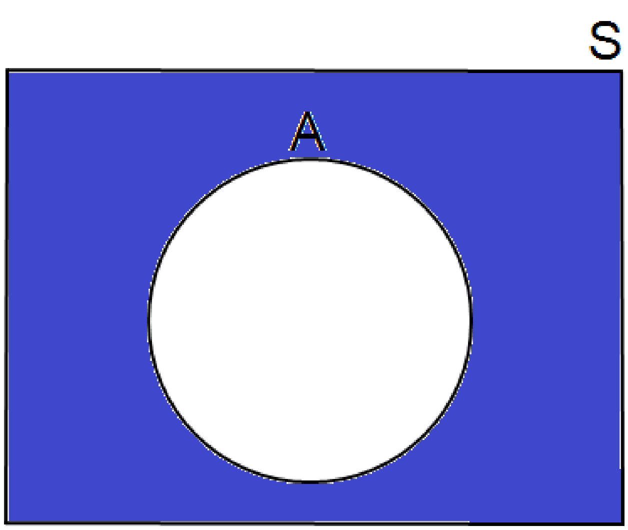
   $P(A) + P(A^C) = P(A \cup A^C) = P(S) = 1$. (3.2)

The figures on the next page show the Venn Diagram representation of the intersection and union of events A and B from the sample space S.





The shaded portion is $A \cup B$.



The shaded portion is A^C.

4. Conditional Probability

1. P(A|B) is the probability that event A will happen given that event B happened.

2. Mathematically, the definition of conditional probability is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \tag{4.1}$$

which implies that $P(A \cap B) = P(B)\,P(A|B)$. (4.2)

5. Probability Independence (Also refer to Page 571—Independent Events)

1. Two sets A and B are independent if

$$P(A \cap B) = P(A)\,P(B), \tag{5.1}$$

which implies that $P(A|B) = P(A)$. (5.2)

[Event B's occurrence does not affect the probability that event A happens.]

To mathematically show that $P(A|B) = P(A)$ is true, we have $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$.

2. Three sets A, B, and C are independent if all four conditions are satisfied:

(a) $P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$

(b) $P(A \cap B) = P(A)\,P(B)$

(c) $P(A \cap C) = P(A)\,P(C)$

(d) $P(B \cap C) = P(B)\,P(C)$

3. If events A, B, and C are independent, all the events that involve the complement, intersection, and union, are independent. i.e. The events such as A, B, C, A^C, B^C, C^C, $A \cap B$, $A \cap B^C$, and $B \cap C^C$ are independent one another. If you are really ambitious, try to show how this works!

6. Probability Dependence

There are incidents that one event affects the other. For example, if we draw 2 cards from a standard deck of cards and want to find the probability that both cards are hearts. Whether the first card is a heart affects the probability that the second card as a heart.

If events A and B are dependent, then $P(A \cap B) = P(B)\,P(A|B)$. This formula is mentioned in the Conditional Probability Section. **In words, if you were to find the probability that both of two dependent events will happen, find the probability that the first event will happen. Assuming that the first event happened (conditional probability), find the probability that the second event will happen. Finally, multiply the two probabilities will get you the answer.**

Same for three events, and so on.

If you were to find the probability that all of three dependent events will happen, find the probability that the first event will happen. Assuming that the first event happened, find the probability that the second event will happen. Assuming that the first two events happened, find the probability that the third event will happen. Finally, multiply the three probabilities will get you the answer.

7. Problems that relate to Probability Theories, Conditional Probability, Probability Independence, and Probability Dependence:

1. If events A and B are independent, P(A) = 0.6, and P(B) = 0.5, what is P(A ∪ B)?

Ans: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (Use Equation 3.1)

$= P(A) + P(B) - P(A)P(B)$ (Use Equation 5.1)

$= 0.6 + 0.5 - (0.6)(0.5)$

$= 0.8$.

2. If P(A) + P(B) = 0.9, P(A|B) = 0.5, P(B|A) = 0.4, what is P(A)?

Ans: By Equation 4.1, P(A|B) = 0.5 implies that $\frac{P(A \cap B)}{P(B)}$ = 0.5, which leads to 0.5P(B) = P(A∩B).

By Equation 4.1, P(B|A) = 0.4 implies that $\frac{P(A \cap B)}{P(A)}$ = 0.4, which leads to 0.4P(A) = P(A∩B).

We see that 0.5P(B) = 0.4P(A), so P(B) = 0.8P(A).

Substituting into P(A) + P(B) = 0.9, we get P(A) + 0.8P(A) = 0.9, or 1.8P(A) = 0.9. So P(A) = 0.5.

3. A bag has 100 marbles that breakdown as follows:

70 red (40 large, 30 small)

30 green (10 large, 20 small)

(a) If we choose one at random and we know the marble is red. What is the probability that it is a large marble?

Ans: Intuitively, because of the 70 marbles, 40 of them are red, the answer is 40/70 = 4/7, but we will show how to get the answer using the concept of conditional probability.

By Equation 4.1, P(large|red) = $\frac{P(large \cap red)}{P(red)} = \frac{40/100}{70/100} = \frac{4}{7}$.

The "100" divides out from both the numerator and the denominator. In this problem, the conditional probability solution is fairly cumbersome, but on some other problems it is useful.

(b) We take out a marble and see that it is a red marble. Without putting it back, we take out another one. What is the probability that the second marble is red?

Ans: After taking out one red marble, the bag has 99 marbles. 69 of the marbles are red. So the probability is 69/99 = 23/33 = P(2nd is red | 1st is red). Using the definition of conditional probability is very annoying. We will skip this by taking this intuitive solution.

(c) Take out two marbles without replacement. What is the probability that both are red?

Ans: P(both red)

= P(1st is red ∩ 2nd is red)

= P(1st is red) P(2nd is red | 1st is red) (Use Equation 4.2)

$= \frac{70}{100} * \frac{69}{99}$.

(d) Take out five marbles without replacement. What is the probability that all of them are large?

Ans: $\frac{50}{100} * \frac{49}{99} * \frac{48}{98} * \frac{47}{97} * \frac{46}{96}$.

Same idea as (c). Writing down the verbal model is very difficult and unnecessary. When you get the idea of the conditional probability and probability dependence, you can immediately come up with the expression that directly leads to the answer.

(e) Choose five marbles with replacement. What is the probability that the marbles are red, red, red, green, red, in this order?

Ans: $\frac{70}{100} * \frac{70}{100} * \frac{70}{100} * \frac{30}{100} * \frac{70}{100}$.

P(red) = 70/100, and P(green) = 30/100. The five selections are independent (the outcome of one selection doesn't affect the other). Hence you can multiply together the probability of each of the 5 events.

(f) Choose five marbles with replacement. What is the probability that at least one marble is red?

Ans: From Equation 3.2, we see that $P(A) = 1 - P(A^C)$.

P(at least one red) = 1 – P(at least one red^C) = 1 – P(no red) = 1 – $(30/100)^5$.

Note: Sometimes thinking about the complement is a much easier way.

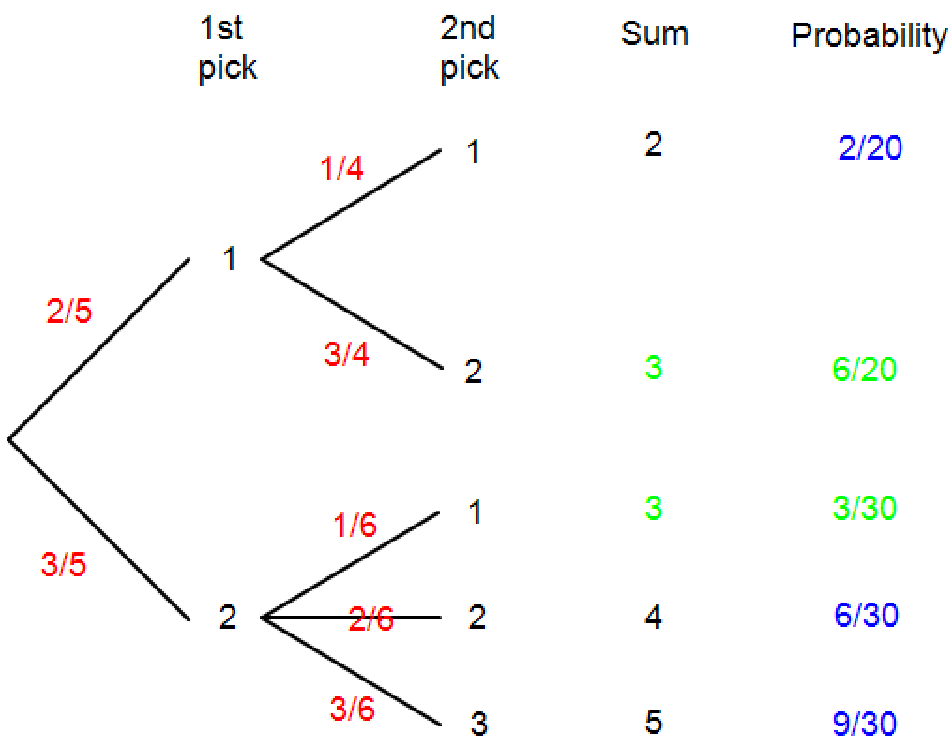
4. Given 2 urns. Urn 1 has two 1's and three 2's. Urn 2 has one 1, two 2's, and three 3's. We randomly choose one number from urn 1 without replacement.

If it is a 1, then we take another number from urn 1.

If it is a 2, then we take number from urn 2.

(a) What is the probability that the sum of the two selected numbers is 3?

Ans: Drawing a tree is a very effective when dealing with these problems.



The red numbers are the probability that each event will happen. Each number in the "Probability" column is the product of the two red numbers in the same branch. i.e. 6/30 = (3/5)(2/6). This concept is related to conditional probability and dependent events.

P(sum is 3) = (6/20) + (3/30) = 0.4.

(b) Given that the sum is 3, what is the probability that the first pick is 2?

$$\text{P(1}^{\text{st}}\text{ pick is 2 | sum is 3)} = \frac{P(1st\ number\ is\ 2 \cap sum\ is\ 3)}{P(sum\ is\ 3)} = \frac{\frac{3}{30}}{0.4} = 0.25.$$

Note: When you are stuck on a probability problem, always draw a tree to break down the problem. You will get some insights from here.

8. Introduction to Factorials (prerequisite for Section 15.3)

Let n be a positive integer. The factorial of n, written as $n!$, is the product of the first n positive integers.

$n! = 1*2*3* \ldots * (n-1) * n$ (8.1)

$1! = 1$

$2! = 1*2 = 2$

$3! = 1*2*3 = 6$

$4! = 1*2*3*4 = 24$

$5! = 1*2*3*4*5 = 120$

. . .

Note the property/definition that $n! = n*(n-1)!$ (8.2)

Ex 1: Simplify $\frac{8!}{6!}$.

$\frac{8!}{6!} = \frac{8*7*6!}{6!} = 8*7 = 56$.

Ex 2: How many ways can 9 people form a line?

According to the multiplication rule (see Page 561), there are 9 possibilities for the 1st position, 8 possibilities for the 2nd position, 7 possibilities for the 3rd position… Thus the answer is $9*8*7*\ldots*2*1 = 9! = 362880$.

Ex 3: How many ways can we arrange n distinguishable objects into a line?

Similar reasoning with Ex 2: the answer is $n!$.

Ex 4: What is 0!?

Interestingly, the answer is 1. Here are two ways to show why this is true.

Method 1: According to Equation 8.2, $n! = n*(n-1)!$. Thus $1! = 1*0!$, which implies that $0! = 1$.

Method 2: (Comes up Ex 2): How many ways can 0 people form a line?

The answer is 1—there is only 1 way—no people are present, and that's the only arrangement.