Duke Math Meet 2017
Relay Round Solution

Relay Round 1

1. We have

$$\begin{aligned}\sin 20^\circ \cdot \sin 10^\circ \cdot \sin 50^\circ \cdot \sin 70^\circ &= \sin 20^\circ \cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ \\ &= \frac{1}{2}\sin 40^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ \\ &= \frac{1}{4}\sin 80^\circ \cdot \cos 80^\circ \\ &= \frac{1}{8}\sin 160^\circ = \frac{1}{8}\sin 20^\circ\end{aligned}$$

Therefore $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{16}$. The answer is $\boxed{16}$

2. $T = 16$. We have $a_{n+1} - 4a_n = 4(a_n - 4a_{n-1}) = 4^2(a_{n-1} - 4a_{n-2}) = \cdots = 4^n(a_1 - 4a_0) = 0$ So we have $a_{n+1} = 4a_n$. Therefore $a_n = 4^n$ and $\log_2 a_{2017} = \boxed{4034}$

3. $T = 4034$. We want to show, by induction, that if there are $n = 2k$ participants, the largest total number of matches played in the tournament is less than k^2. First, when $k = 1$, there are only two participants, so the total number of matches can't exceed 1. Therefore the statement is true. Suppose it is true for $k-1$, then given a tournament with $2k$ participants, we can find a pair that has played a match. Let's say participant A played with participant B. If they together played more than $n-1$ matches in total, then by pigeonhole principle, there exists another participant C who played with both A and B. It contradicts to the fact that no 3 participants of whom each pair has played with each other. So the total number of matches played by A and B is at most $n-1$. The rest $2n-2$ participants played at most $(k-1)^2$ matches by induction hypothesis. Therefore the total number of matches in this tournament is at most $(k-1)^2 + n - 1 = (k-1)^2 + 2k - 1 = k^2$. Taking $n = 4034 = 2k$, we obtain the answer $\boxed{2017^2 = 4068289}$

Relay Round 2

1. We have $p = (c^2 - q)(c^2 + q)$. Since p is a prime, $c^2 - q = 1$ and $c^2 + q = p$. Now we know $q = c^2 - 1 = (c-1)(c+1)$. So either $c - 1 = 1$ and $c + 1 = q$ or $c - 1 = -q$ and $c + 1 = -1$. Both solutions show that $q = 3$. Then $p = 7$. Hence $p + q = \boxed{10}$

2. $T = 10$. Let s_n be the number of subsets in $\{1, 2, ..., n\}$ that do not contain two consecutive numbers. Then $s_n = s_{n-1} + s_{n-2}$. We know $s_1 = 2$ and $s_2 = 3$. So $s_{10} = \boxed{144}$

3. $T = 144$. Since $a + b + c = 0$, $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc) = 0$. Hence, $abc = \boxed{\frac{e^{-6}}{3}}$.