Power Round: Ramsey Theory

Duke Math Meet 2017

During this round, we will explore some results in Ramsey theory. Some of the later problems may need earlier results, so feel free to use any previous parts even if you are not able to solve them yet. You can also use any given theorems or definitions in this handout.

We will assume that the natural numbers ($\mathbb{N}$) start with 1.

1 Introduction

Problem 1. (2 points) There are six people at a party. We assume that for every pair of them, they are either friends or not friends (i.e. strangers). Prove that either there are three people all of whom are friends, or there are three people of whom no two are friends.

(Hint: show that there is someone who has either three friends or three strangers)

This result is a special case of a theorem published by Ramsey in 1930. The original theorems of Ramsey have been extended into many directions, resulting what is now known as *Ramsey Theory*. This famous "party theorem" highlights the flavor of Ramsey theory - the idea that some patterns are unavoidable when the structure is large enough. Another well-known example is the pigeonhole principle, which states that no matter how we partition a set of $kn+1$ elements into k subsets, there exist one subset that has at least n elements. In this handout, we aim to examine some similar results of this type and in particular we are going to study Ramsey numbers.

The generalizations of this problem are easier to formulate with Graph Theory. So let's first introduce some basic concepts.

Definition 1.1. A *graph* $G = (V, E)$ is an ordered pair such that V is a set of elements, usually called *vertices* or *nodes*, and $E \subseteq \{\{u, v\}|\ u, v \in V,\ u \neq v\}$ is the set of *edges*.

Definition 1.2. We say that $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

Definition 1.3. An r*-coloring* on the edges of a graph $G = (V, E)$ is a coloring function $c : E \to \{1, 2, .., r\}$, i.e. we are assigning each edge one of the r colors. When $r = 2$, we usually use the term "red-blue coloring" with the coloring function $c : E \to \{\text{red}, \text{blue}\}$.

Definition 1.4. A *complete graph* $K_n = (V, E)$ is graph such that $|V| = n$ and $E = \{\{u, v\}|u, v \in V,\ u \neq v\}$.[1] In other words, K_n is a graph with n vertices and every pair of its vertices has an edge.

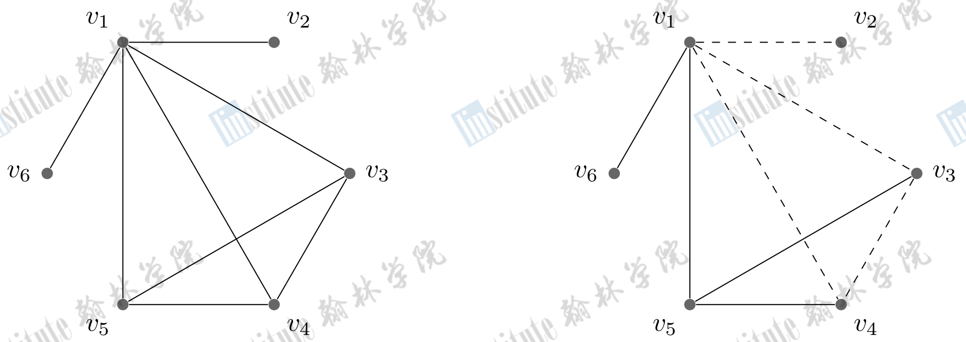
Definition 1.5. Let $K_n = (V, E)$ be a complete subgraph of G and c be a r-coloring on the edges of G, we say K_n is *of color* j if $c(e) = j$ for all $e \in E$. We say K_n is *monochromatic* if there exists a color $j \in \{1, 2, ..., r\}$ such that K_n is of color j.

[1] Throughout this handout, we will use the notation $|S|$ to denote the number of elements in the set S

Example 1.6. Let $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{\{v_1, v_2\},\ \{v_1, v_3\},\ \{v_1, v_4\},\ \{v_1, v_5\},\ \{v_1, v_6\},\ \{v_3, v_4\},\ \{v_3, v_5\},\ \{v_4, v_5\}\}$. Usually we can picture a graph by drawing vertices and edges. For instance, figure 1a gives us a visualization of graph G.

Example 1.7. Let G be the graph defined in example 1.6 and $G' = (V', E')$ with $V' = \{v_1, v_3, v_4, v_5\} \subseteq V$ and $E' = \{\{v_1, v_3\},\ \{v_1, v_4\},\ \{v_1, v_5\},\ \{v_3, v_4\},\ \{v_3, v_5\},\ \{v_4, v_5\}\} \subseteq E$. Then G' is a subgraph of G.

Example 1.8. G' defined in example 1.7 is a complete graph with four vertices. Suppose c is a red-blue coloring on the edges of G with $c(\{v_1, v_2\}) = c(\{v_1, v_3\}) = c(\{v_1, v_4\}) = c(\{v_3, v_4\}) =$ red and $c(\{v_1, v_5\}) = c(\{v_1, v_6\}) = c(\{v_4, v_5\}) = c(\{v_4, v_5\}) =$ blue. Then we can picture this coloring with figure 1b.



(a) Graph $G = (V, E)$

(b) Red-blue coloring on G.
(red: dashed lines; blue: solid lines)

Figure 1: Example 1.6, 1.7 and 1.8

Theorem 1.9. *For any red-blue coloring on a complete graph K_6, there exists a monochromatic subgraph K_3.*

Theorem 1.9 is a rephrase of problem 1 with graph theory. Indeed, suppose those six people form a vertex set of a complete graph. If two people are friends with each other, we color the edge connecting them with red, otherwise we color it with blue. Problem 1 simply says that there's a group of three people such that the edges connecting them are of the same color. Then this group forms a monochromatic subgraph K_3.

Problem 2. (1 point) Suppose V is a vertex set with $|V| = n$ and $K_n = (V, E_K)$ is the complete graph on the vertex set V. Show that there's a one-to-one correspondence between subgraphs $G = (V, E \subseteq E_K)$ of K_n on the vertex set V and 2-coloring functions c on the edges of K_n.

2 The Ramsey Numbers

Definition 2.1. Given $s, t \in \mathbb{N}$. the *Ramsey number* $R(s, t)$ is the smallest value of $n \in \mathbb{N}$ such that every red-blue coloring of a complete graph K_n yields either a red subgraph K_s or a blue subgraph K_t.

Here are some basic properties of Ramsey numbers.

Problem 3. Given $s, t \in \mathbb{N}$. Prove the following properties:

(i) (1 point) $R(2, s) = s$.

(ii) (2 points) $R(s, t) = R(t, s)$.

Problem 4.

(i) (1 point) Show that there exists a red-blue coloring on the edges of K_5 that contains no monochromatic K_3.

(ii) (1 point) Show that $R(3, 3) = 6$.

3 Bounds on Ramsey Numbers

Although no formula has been found for general Ramsey numbers, the bounds on Ramsey numbers are well-studied. The following result by Erdos and Szekeres gives an upper bound on Ramsey numbers.

Problem 5. Given $s, t \in \mathbb{N}$.

(i) (1 point) If $R(s, t) \leq N$ for some positive integer N. Show that for any two coloring of K_N, there exists either a red K_s or a blue K_t.

(ii) (2 points) Assuming $R(s-1, t)$ and $R(s, t-1)$ are finite, show that

$$R(s, t) \leq R(s-1, t) + R(s, t-1)$$

(iii) (1 point) Prove by induction that $R(s, t)$ is finite for all $s, t \in \mathbb{N}$.

(iv) (2 points) Prove that[2]

$$R(s, t) \leq \binom{s+t-2}{s-1}$$

(v) (1 point) Show that

$$R(s, s) \leq 2^{2s-2}$$

4 Exact Ramsey Numbers $R(s, t)$ for Small s and t

We showed in problem 3 that $R(2, s) = s$ for all $s \in \mathbb{N}$. It is natural to think about $R(3, s)$. The following problem establish a lower bound for such Ramsey numbers.

Problem 6.

(i) (2 points) In figure 2, we view solid lines as red edges and view all pairs of nodes that are not connected as blue edges. Then this graph shows that $R(3, 4) > 8$. By extending this construction, show that $R(3, s+1) > 3s - 1$ for all integers $s \geq 2$.

[2] We use the notation $\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$ for binomial coefficient. $0! = 1$ by convention.

(ii) (2 points) Show that $R(3,4) = 9$.

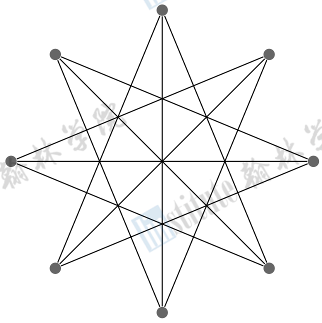


Figure 2: A graph shows that $R(3,4) > 8$

Problem 7. (2 points) By considering the graph with vertex set $\mathbb{Z}_{17}$ (the integers modulo 17) in which the pair (i,j) is connected by an edge if and only if

$$i - j = \pm 1,\ \pm 2,\ \pm 4,\ \pm 8 \pmod{17},$$

show that $R(4,4) = 18$.

5 Generalized Ramsey Numbers for r-coloring

Definition 5.1. Given $r \geq 2$ and $s_1,\ s_2,\ ...,\ s_r \in \mathbb{N}$. The generalized Ramsey number $R_r(s_1, s_2, ..., s_r)$ is the smallest value of n such that for any r-coloring on the edges of K_n, there's a complete subgraph K_{s_j} of color j for some color $j \in \{1, 2, ..., r\}$.

By definition, we have $R_2(s_1, s_2) = R(s_1, s_2)$.

Problem 8.

(i) (2 points) Show that

$$R_r(s_1, s_2, ..., s_r) \leq R_{r-1}(R(s_1, s_2), s_3, ..., s_r)$$

(ii) (2 points) Find an upper bound for $R_3(3,3,3)$ and justify it.