Duke Math Meet 2016

Tiebreaker Solutions

1. The equation has roots if $(b + 2n)^2 - 4(a + n)(c + n) = b^2 - 4ac + 4n(b - a - c) \geq 0$. If $b - a - c \geq 0$ then $b^2 \geq (a + c)^2 \geq 4ac$. If $b - a - c < 0$, then for some large enough n, then $b^2 - 4ac + 4n(b - a - c) < 0$. Hence it is enough to find $b \geq a + c$.

   The number of solutions $a+c \leq i$ is equal to the number of solutions to $a'+c'+d' = i-2$ where $a', c', d' \geq 0$ which is $\binom{i}{2} = \frac{i(i-1)}{2}$. $\sum_{i=1}^{10} \frac{i(i-1)}{2} = \boxed{165}$.

2. We can bound the n by using minimum area. So we have $1(2) + 3(4) + 5(6) + 7(8) + 9(10) = 190$ and $1(10) + 2(9) + 3(8) + 4(7) + 5(6) = 110$. So $11 \leq n \leq 13$. Using a rotating flower shape, we can see that $\boxed{11}$ works.

3. Call the point B the point on the original circle at which the aircraft is positioned when the missle is fired. We claim that the path of the missile is a circle with radius that has AB tangent to it. Let P be some arbitrary point along the path of the aircraft. Call the intersection of PA with the new circle be point M. Then $\angle PAB$ is half the measure of the arc MA. Since the missile and aircraft is the same speed, they should travel equal distance in equal times, so $PB = AM$. Since the measure of PB is the measure of $\angle PAB$, the radius of the smaller circle is half the radius of the larger. Hence the missile travels half the circumference of the circle or $\boxed{6\pi}$.