

1. a. $\frac{41}{24}$. Going reverse. $2 + \frac{1}{3} = \frac{7}{3}, 2 + \frac{3}{7} = \frac{17}{7}, 1 + \frac{7}{17} = \frac{24}{17}, 1 + \frac{17}{24} = \frac{41}{24}$
- b. $[1, 2, 3, 4] \frac{43}{30} = 1 + \frac{13}{30}, \frac{30}{13} = 2 + \frac{4}{13}, \frac{13}{4} = 3 + \frac{1}{4}, 4 = 4 + 0,$
- c. $[1, \overline{1, 2}]$

$$\begin{aligned}\sqrt{3} &= 1 + (\sqrt{3} - 1) \\ \frac{1}{\sqrt{3} - 1} &= \frac{\sqrt{3} + 1}{2} \\ \frac{\sqrt{3} + 1}{2} &= 1 + \frac{\sqrt{3} - 1}{2} \\ \sqrt{3} + 1 &= \frac{2}{\sqrt{3} - 1} \\ \sqrt{3} + 1 &= 2 + \sqrt{3} - 1\end{aligned}$$

Here we note that, we already encountered $\sqrt{3} - 1$ (remainder of the first step) so we see that continued fraction will repeat.

- d. $\sqrt{6}$

Let $x = [2, 4]$. Then we have $x = [2, 4, x]$

$$\begin{aligned}x &= [2, 4, x] \\ x &= [2, 4 + \frac{1}{x}] \\ x &= [2, \frac{4x + 1}{x}] \\ x &= 2 + \frac{x}{4x + 1} \\ x &= \frac{9x + 2}{4x + 1}\end{aligned}$$

So we see that $4x^2 + x = 9x + 2 \implies 2x^2 - 4x - 1 = 0$. So $x = \frac{4 \pm 2\sqrt{6}}{4} = \frac{2 \pm \sqrt{6}}{2}$. Remember that we want $2 + \frac{1}{x} = 2 + \sqrt{6} - 2 = \sqrt{6}$.

2. a. We prove by induction. Base case: $n = 1$. $p_1q_0 - p_0q_1 = (a_1a_0 + 1)(1) - a_0a_1 = 1 = (-1)^0$. Now suppose the claim is true for n .

$$\begin{aligned}p_{n+1}q_n - p_nq_{n+1} &= (a_{n+1}p_n + p_{n-1})q_n - p_n(a_{n+1}q_n + q_{n-1}) \\ &= p_{n-1}q_n - p_nq_{n-1} \\ &= (-1)(-1)^{n-1} \\ &= (-1)^n\end{aligned}$$

- b. Note we used the identity from part a.

$$\begin{aligned}p_{n+1}q_{n-1} - p_{n-1}q_{n+1} &= (a_n p_n + p_{n-1})q_{n-1} - p_{n-1}(a_n q_n + q_{n-1}) \\ &= a_n(p_n q_{n-1} - p_{n-1} q_n) \\ &= a_n(-1)^{n-2} \\ &= (-1)a_n\end{aligned}$$

3. We first find the continued fraction of $\frac{1+\sqrt{5}}{2}$.

$$\begin{aligned}\frac{1+\sqrt{5}}{2} &= 1 + \frac{\sqrt{5}-1}{2} \\ \frac{1+\sqrt{5}}{2} &= \frac{2}{\sqrt{5}-1}\end{aligned}$$

So $\frac{1+\sqrt{5}}{2} = [1]$. Hence all the $a_i = 1$.

Note that we have $p_n = p_{n-1} + p_{n-2}$ and $p_0 = 1 = F_1, p_1 = 2 = F_2$. Thus $p_n = F_{n+1}$. We also have $q_n = q_{n-1} + q_{n-2}$ and $q_0 = 1 = F_0, q_1 = 1 = F_1 \implies q_n = F_n$. So we have the n th convergent $\frac{p_{n-1}}{q_{n-1}} = \frac{F_n}{F_{n-1}}$

4. $(-34, 74)$ or anything of the form $(-34 + 567t, 74 - 1234t)$. We first find the continued fraction expansion $\frac{1234}{567} = [2, 5, 1, 2, 33]$.

$$\begin{array}{c|c|c|c|c} a_n & 2 & 5 & 1 & 2 & 33 \\ p_n & 2 & 11 & 13 & 37 & 1234 \\ q_n & 1 & 5 & 6 & 17 & 567 \end{array}$$

From problem we have that $1234(17) - 567(37) = -1$. So $1234(-34) + 567(74) = 2$.

5. a. From the given theorem, we know that α lies between any two consecutive convergents. So for $n \geq 1$, we have $|\alpha - \frac{p_n}{q_n}| < |\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}| < \frac{1}{q_{n+1}q_n}$.

b. Suppose not. Then there exists n such that $|\alpha - \frac{p_n}{q_n}| < \frac{1}{2q_n^2}$ and $|\alpha - \frac{p_{n+1}}{q_{n+1}}| < \frac{1}{2q_{n+1}^2}$. $|\alpha - \frac{p_n}{q_n}| + |\alpha - \frac{p_{n+1}}{q_{n+1}}| \geq |\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}| = \frac{1}{q_nq_{n+1}}$. But $\frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2} \geq \frac{1}{q_nq_{n+1}}$ by AM-GM inequality. Contradiction. Thus at least one of any two consecutive convergents to x works.

c. We can see that $\sqrt{2} = [1, \bar{2}]$.

$$\begin{array}{c|c|c|c|c|c} a_n & 1 & 2 & 2 & 2 & 2 \\ p_n & 1 & 3 & 7 & 17 & 41 \\ q_n & 1 & 2 & 5 & 12 & 29 \end{array}$$

$29 * 70 > 1000$ so we can use $\frac{41}{29}$ or $\frac{99}{70}$.

6. a. $\alpha - \alpha' > 0 \implies \frac{\sqrt{D}}{Q} > 0$ so $Q > 0$.

$\alpha + \alpha' > 0$, so $P > 0$.

$\alpha' < 0$ so $P < \sqrt{D}$

$\alpha > 1$ so $Q < P + \sqrt{D} < 2\sqrt{D}$

b. $\alpha - [\alpha]$ is of the form $\frac{A+\sqrt{D}}{B}$. Then $\frac{B}{A+\sqrt{D}} = \frac{-BA+\sqrt{D}B}{(D^2-A)}$. So α_1 is a quadratic irrational. Since $\alpha - [\alpha] < 1$, we know that $\alpha_1 > 1$. In addition, note that $\alpha' = [\alpha] + \frac{1}{\alpha'_1} \implies \alpha'_1 = -\frac{1}{q_0 - \alpha'}$. α' is negative and q_0 is positive so $-1 < \alpha'_1 < 0$. So α_1 is reduced.

c. We know that inductively all the α_i are reduced. But there are a finite number of possibilities as seen by problem 6. Thus there exists $i < j$ such that $\alpha_i = \alpha_j$, and so $\alpha_{i+1} = \alpha_{j+1}$ and so on. So the sequence is periodic.

7. a. We want $\sqrt{N} + m > 1$ and $-1 < m - \sqrt{N} < 0$. So we see that $m = [\sqrt{N}]$.

b. We know that $\sqrt{N} + m = [a_0, \dots, a_n]$. But note that $a_0 = [\sqrt{N} + m] = 2m$. So we have $\sqrt{N} + m = [2m, \dots, a_n]$ so $\sqrt{N} = [m, a_1, \dots, a_n, 2m]$.

8. a. The area would be $\frac{1}{2}|P_{n+2}Q_n - Q_{n+2}P_n| = \frac{a_{n+2}}{2}$ since triangle is half of the parallelogram. However since $P_{n+2} = a_{n+2}P_{n+1} + P_n$ and $Q_{n+2} = a_{n+2}Q_{n+1} + Q_n$. $(P_n + Q_n) + i(P_{n+1}, Q_{n+1})$ for $0 \leq i \leq a_{n+2}$ must all be boundary points. So we have $a_{n+2} + 2$ boundary points (don't forget to include origin). So by Pick's, we see that $I = 0$.

(a) Area of parallelogram is $|2PQ - 2Q^2\alpha| = 2Q^2|\alpha - \frac{P}{Q}| < 1$. Now suppose there was an interior point in the region (x, y) , then $(2P - x, 2Q - y)$ is also in the region. But then consider the figure $(0, 0), (x, y), (P, Q)$, and $(2P - x, 2Q - y)$. Pick's Theorem tells us that this area is at least 1. But this lies in the parallelogram. Contradiction.

c. Suppose that $|\alpha - \frac{P}{Q}| < \frac{1}{2Q^2}$ and $\frac{P}{Q}$ isn't a continued fraction convergent. Then from part b, we know no lattice points in that parallelogram. But then (P, Q) lies in the triangle of $(0, 0), (P_n, Q_n)$, and (P_{n+2}, Q_{n+2}) . This can be seen as all three points lie on the same side of line thru $(2Q\alpha, 2Q)$ since by considering slope, we know (P_n, Q_n) and (P_{n+2}, Q_{n+2}) are on the same side and (P, Q) is between those two and thus must be on the same side. Contradiction to part a.