Duke Math Meet
Power Round

November 19th,2005

There are multiple problems related to a common theme. Some of the later questions may require or be supplemented by the earlier questions. All answers must be in the form of a complete solution with proof.

Commuting Polynomials

It is well known that composition of polynomials is not, in general, commutative. For instance, if $f(x) = 2x$ and $g(x) = x + 1$, then $f(g(x)) = 2x + 2$, but $g(f(x)) = 2x + 1$. However there are some polynomials which do commute; for instance, if $f(x) = x + 3$ and $g(x) = x + 4$ then $f(g(x)) = g(f(x)) = x + 7$. In this round, we will investigate some special cases of polynomials that commute.

1. Let $f(x) = ax + b$ with $a \neq 0$ and b real numbers. Find all pairs of real constants c and d with $c \neq 0$ such that $g(x) = cx + d$ commutes with $f(x)$, that is $f(g(x)) = g(f(x))$.
2. Let $f(x) = ax^b$ with $a \neq 0$ real and b a positive integer. Find all pairs of real numbers d and c with $c \neq 0$ such that $g(x) = cx^d$ commutes with $f(x)$.
3. Let $f(x) = ax + b$ with a and b real numbers, $a \neq 0$.
   (a) Show that if $f(x)$ commutes with a polynomial $g(x)$ with degree of $g \geq 2$, then $a = \pm 1$.
   (b) Take $g(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $a_n \neq 0$ and $n \geq 2$. For $a = 1$, show that if $f(x)$ commutes with $g(x)$ then $b = 0$ so $f(x) = x$.
   (c) For $a = -1$, again show that if $f(x)$ commutes with $g(x)$ then $b = 0$ so $f(x) = -x$. Describe all polynomials which commute with $f(x) = -x$. Conclude that the only linear polynomial functions which commute with higher degree polynomials are x and $-x$.
4. Let $f(x) = x + 2\pi$.
   (a) Find a non-polynomial function that commutes with $f(x)$.
   (b) Show that there are infinitely many functions that commute with $f(x)$.
5. Let $\{T_n(x)\}_{n=0}^{\infty}$ be an infinite sequence of polynomials T_n defined by $T_0(x) = 1$, $T_1(x) = x$, and $T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$ for $n \geq 0$.
   (a) Find $T_2(x)$ and $T_3(x)$.
   (b) Show that $T_2(x)$ and $T_3(x)$ commute.
   (c) Show that $\pm T_n(x)$ and $\pm T_m(x)$ commute.
6. Denote by $R_q^{(l)}$ the polynomial defined by $R_q^{(1)} = xP(x^q)$, $R_q^{(n+1)}(x) = R_q^{(1)}(R_q^{(n)}(x))$ for $n > 1$, where $P(x)$ is an arbitrary fixed polynomial.
   (a) If $q = 2$, find suitable constants $c_1 \neq 1$ and $c_2 \neq 1$ such that $c_1 R_2^{(l)}(x)$ commutes with $c_2 R_2^{(m)}(x)$.
   (b) In the general case, find all constants $c_1$and c_2 such that $c_1 R_q^{(l)}(x)$ commutes with $c_2 R_q^{(m)}(x)$.
7. Let $f_m(x) = x^m$.
   (a) Find all polynomials $g(x)$ of degree greater than two that commute with $f_2(x)$.
   (b) Find all polynomials $g(x)$ of degree greater than two that commute with $f_m(x)$.